Introductory Notes on Predicate Logic[*]

Michael E. Cuffaro

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[*] These notes are drawn substantially from: Merrie Bergmann, James Moor, Jack Nelson, "The Logic Book (5th edition)", McGraw Hill, 2009.

1 Introduction to Predicate Logic

1.1 Advantages and Disadvantages of Sentential Logic

Advantages

1. For english arguments, sentences, and sets of sentences that have been symbolised adequately, *Truth functional* validity, truth/falsity, (in)consistency in SL yields *logical* validity, truth/falsity, (in)consistencyin English.
2. Table and tree methods (which determine these truth-functional properties are mechanical procedures which always end with a yes/no answer after a finite number of steps. Therefore truth-functional validity, truth/falsity, (in)consistency, equivalence, entailment are *decidable properties*.

Major Disadvantage

- SL is not sophisticated enough to symbolise most of natural language.

E.g., a valid argument in English:

All whales are mammals.

Moby Dick is a whale.

Therefore Moby Dick is a mammal.

Symbolised in SL:

A

D

—

M

Invalid!

Nonetheless there are uses of SL.

- E.g. Computer circuits. (These do not need to replicate natural language).

1.2 Advantages and Disadvantages of Predicate Logic

Advantage

- Much more sophisticated than SL. Can capture much (but not all) of natural language.

Disadvantage

- Not decidable.

1.3 Predicates

Logical predicates are similar (but not identical) to grammatical predicates. They tell you something about the subject(s) of a sentence.

Examples:

Socrates is a man.

- Subject: Socrates
- Predicate: ___ is a man.

This predicate tells you that whatever you fill in the space with is a man. Depending on what goes in the 'subject' space, the sentece can be either true or false.

- Stephen Harper is a man ⇒ True.
- Miley Cyrus is a man ⇒ False.

1.3.1 n-place Predicates

Consider: “Jim is taller than Mary”

There are two ways to break this sentence up into subject/predicate:

1. 1-place Predicate: “___ is taller than Mary”
2. 2-place Predicate: “___ is taller than ___”

In general,

- One space: 1-place predicate
- Two spaces: 2-place predicate
- Three spaces: 3-place predicate
- Four spaces: 4-place predicate
- etc.

In general:

- n-spaces: n-place predicate.

3-place predicate example: “___ gives ___ to ___”

Note: a predicate is not a sentence. Sentences never have blanks. Predicates express relations between the blanks.

1.4 Variables

When we specify predicates, it is usually more convenient if we use variables instead of spaces.

We use the letters: w,x,y,z.

1-place:

- “x is beautiful” instead of “___ is beautiful”
- shortened version: **Bx**

2-place:

- Nxy: x is north of y.

3-place:

- Bxy: x is between y and z

Variables are arbitrary:

- “x is between y and z” means the same as “y is between z and x”
- They both say that something is between something else and some other thing.
- However, be careful: they do not mean the same as “x is between x and y”. This says that x is between itself and something else.

1.5 Singular terms

Singular terms fill in the spaces/variables in a predicate.

Singular terms pick out *one thing.*

Two types:

1. Proper names: e.g., Bertrand Russell, Miley Cyrus
2. Definite Descriptions: e.g., “the third planet from the sun”
    - attempt to pick out one thing in virtue of grammatical structure (usually involve the word “the”).

Singular terms are represented by **individual constants**: a,b,c,d,...v

1.6 Symbolization key

Defines:

- Universe of Discourse (the *things* we are discussing).
- Predicates
- Individual constants representing singular terms

E.g.
UD: Students at UWO
Lxy: x loves y.
Txy: x texts y.
a: Angie
b: Bob
c: Clara

Sentences using this symbolization key:

Lbc: Bob loves Clara.

Lcb: Clara loves Bob.

Lbb: Bob loves himself.

Lab & ∼Tab: Angie loves Bob but does not text him.

Lcb ⊃∼Lbc: Clara loves Bob only if he does not love her.

1.7 Quantifiers

Consider:
UD: People in Bob's car (4-seater)
Dx: x is a driver.
Px: x is a passenger
a: Andy
b: Bob
c: Clara
d: Denise

How do we symbolise “Everyone is either a passenger or a driver”?

$$[(Pa \vee Da) \,\&\, (Pb \vee Db)] \,\&\, [(Pc \vee Dc) \,\&\, (Pd \vee Dd)]$$

This works well for small UDs, but what if we change the symbolisation key:
UD: Everything on Earth
Dx: x is a driver.
Px: x is a passenger
a: Andy
b: Andy’s basketball
c: the fungus on Jim’s apple
d: Sam’s paint brush
e: my right shoelace
and so on ...

$$([(Pa \vee Da) \,\&\, (Pb \vee Db)] \,\&\, [(Pc \vee Dc) \,\&\, (Pd \vee Dd)]) \,\&\, [(Pe \vee De)...$$

Will this sentence ever end? Maybe...

What if we change the UD to be the *positive integers.*

Positive integers: {1,2,3,...} is an infinite set.

So we can’t represent a sentence of the form “Every positive integer is such-and-such” with a finite sentence using this method.

And PL does not allow us to use infinite sentences.

The solution: quantifiers

$\forall$ (universal): translates as “all”, “every”, “for every”, etc.

- Means: everything in the UD.

$\exists$ (existential): translates as “some”, “a”, “there is at least one”, etc.

- Means: there is something in the UD.

We write: $(\forall y)Py$, $(\exists x)(Py \,\&\, Qy)$, etc. Quantifiers are always enclosed in parentheses.

UD: Positive integers
Ox: x is odd
Ex: x is even
Sx: x has a successor[1]
Dxy: x is divisible by y
Gxy: x is greater than y
Lxy: x is less than y
Zx: x is greater than zero
a: 1
b: 2
c: 3

Translating from PL to english:

$(\forall y)Sy \Rightarrow$ "for every y in the UD, y has a successor"'

Since the UD is positive integers, we can say: "every positive integer has a successor"

$(\exists z)Ez \Rightarrow$ "there is at least one member of the UD, z, such that z is even"

Or less formally: "some positive integers are even" (Note: for our purposes, "some" and "there is at least one" mean the *same thing*).

Translating from english to PL:

"Every positive integer is divisible by 1".

Since the UD is the positive integers we can rephrase this as: "For every x in the UD, x is divisible by 1"

- $(\forall x)Dxa$

"Not all positive integers are greater than 2"

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[1] The successor of a number is the number that is one greater than it. E.g. the successor of 5 is 6; the successor of 192 is 193, etc.

This can be rephrased as "It is not the case that for every w in the UD, that w is greater than 2"

- $\sim (\forall w)Gwb$

"Some positive integers are less than 3" ⇒ "There is at least one member of the UD, x, such that x is less than 3"

- $(\exists x)Lxc$

"No positive integer is less than 1"

No, in this case, means **there aren't any** positive integers that have this property. So we can rephrase this as: "It is not the case that there is a member of the UD, x, such that x is less than 1"

- $\sim (\exists x)Lxa$

1.8 Equivalences

$\sim (\exists x)Ax$ means the same as $(\forall x) \sim Ax$

- "It is not the case that there is a member of the UD, x, such that it is an A" *means the same as* "Every member, x, of the UD is such that it is not an A"

$\sim (\forall x)Ax$ means the same as $(\exists x) \sim Ax$

- "It is not the case that every member, x, of the UD is such that it is an A" *means the same as* "there is at least one member of the UD, x, such that it is not an A"

1.9 Truth-functional compounds of quantified sentences

Examples:

“If every positive integer is greater than 1, then there is a positive integer such that 1 is less than it”

- $(\forall x)Gxa \supset (\exists x)Lax$

“If 1 is divisible by itself, then so is every other positive integer”

- $Daa \supset (\forall x)Dxx$

“Not every positive integer is greater than 1, but every positive integer is greater than 0”

- $\sim (\forall x)Gxa \ \& \ (\forall x)Zx$

“If every positive integer is greater than 0, then 3 is.”

- $(\forall w)Zw \supset Zc$

2 Formal Syntax

2.1 Review of metavariables:

P, Q, R: placeholders for expressions.

- Can be filled in with any expression.
- E.g. $\mathbf{P} \supset \mathbf{R}$, can replace $\mathbf{P}$ with $(A \vee B)$ and $\mathbf{Q}$ with $(C \supset (D \vee E))$:
- becomes $(A \vee B) \supset (C \supset (D \vee E))$

a: metavariable for individual constants.

- $(\forall \mathbf{a})$, $(\exists \mathbf{a})$, etc.
- Can be filled in with any constant.

x: metavariable for individual variables.

- $(\forall \mathbf{x})$, $(\exists \mathbf{x})$, etc.
- Can be filled in with any variable.

2.2 Expressions of PL

An expression is any sequence of (not necessarily distinct) elements of the vocabulary of PL.

Valid expression: ())PQPFaGyyzxHv&)C

- Obviously this is nonsense, but it is an expression nevertheless.

Not an expression: P∨3Xar

- 3 is not a part of the vocabulary of PL.

2.3 PL Vocabulary

Punctutation symbols: (,)

Individual Term symbols

- constant symbols: a,b,c,...,v
- variable symbols: w,x,y,z,...

Predicate Symbols: A, B, C, D, ...

T-F connective symbols: $\sim, \&, \vee, \supset, \equiv$

Quantifier symbols: $\forall, \exists$

2.4 Logical operators

T-F Connectives: $\sim, \&, \vee, \supset, \equiv$

Quantifiers

- a more complex type of expression.
- use (but are not the same as) quantifier symbols
- of the form: $(\forall x)$, $(\exists y)$
- quantifiers "contain" variables, e.g.: $(\forall y)$ contains y.

2.5 Formulas

Expressions that meet certain requirements are called formulas.

- Atomic formulas
- Truth-functional compounds
- Quantified formulas

2.5.1 Atomic formulas:

- Sentence letters (like the sentences of SL. Can think of these as 0-place predicates)
- n-place predicates (Px, Qa, Rax, Vxy, Bab, etc.)

2.5.2 Truth-functional compounds

- T-F connective is the main connective.
- e.g. $\sim A$, $A \,\&\, B$, $B \vee (C \,\&\, D)$, etc.

2.5.3 Quantified formulas

- Quantifier is the main connective.
- e.g. $(\forall x)Ax$, $(\exists x)Ax$.

2.5.4 Scope of a connective in general

- What a connective ranges over.
- Includes the connective itself.
- E.g. $A \supset (B \vee C)$
  - Scope of $\vee$ is $(B \vee C)$
  - Scope of $\supset$ is $A \supset (B \vee C)$
- Main connective: the connective whose scope is the whole formula (in this case, $\supset$).

2.5.5 Scope of a quantifier

- Quantifier is a unary operator. It is similar to $\sim$.
- The scope of $\sim$ in the formula $Ax \supset\sim (Bx \vee Cx)$ is $\sim (Bx \vee Cx)$.
- Likewise the scope of $(\forall x)$ in $Ax \supset (\forall x)(Bx \vee Cx)$ is $(\forall x)(Bx \vee Cx)$.

2.6 General definition of a formula:

A: An atomic formula is a formula.

B: $\sim\mathbf{P}$ is a formula if $\mathbf{P}$ is a formula.

C: $(\mathbf{P}\ \&\ \mathbf{Q})$, $(\mathbf{P} \vee \mathbf{Q})$, $(\mathbf{P} \supset \mathbf{Q})$, $(\mathbf{P} \equiv \mathbf{Q})$ are formulas if both $\mathbf{P}$ and $\mathbf{Q}$ are formulas.

D: $(\forall \mathbf{x})\mathbf{P}$ and $(\exists \mathbf{x})\mathbf{P}$ are formulas if $\mathbf{P}$:

- is a formula.
- contains at least one occurrence of $\mathbf{x}$.
- does not already contain an $\mathbf{x}$-quantifier.

E: Only expressions conforming to A-D are formulas.

2.6.1 Stepwise Procedure for determining whether an expression P is a formula

1. Is the expression of form A (i.e. an atomic formula)?
    - If yes, then you are done. the expression is a formula.
2. Is the expression of the form B, C, or D?
    - If no, then you are done. the expression is not a formula.
    - If yes, then go back to step 1 for each sub-expression of the expression.

Example: Raxyb

- Is Raxyb an atomic formula?
- Yes! we are done.

Example: $\sim (\forall x)(Ax \supset (Bx \vee Cx)$

- Is it an atomic formula? No.
- Is it of the form $\sim\mathbf{P}$? Yes. So it is a formula if **P** is a formula. Go back to step 1 for the sub-expression $(\forall x)(Ax \supset (Bx \vee Cx)$
- Is it an atomic formula? No.
- Is it of the form $\sim\mathbf{P}$? No.
- Is it of the form $(\mathbf{P}\ \&\ \mathbf{Q})$, $(\mathbf{P} \vee \mathbf{Q})$, $(\mathbf{P} \supset \mathbf{Q})$, or $(\mathbf{P} \equiv \mathbf{Q})$? No.
- Is it of the form $(\forall\mathbf{x})\mathbf{P}$ or $(\exists\mathbf{x})\mathbf{P}$? Yes!
- Does the subexpression contain at least one occurrence of **x**? Yes. Does it contain an **x**-quantifier? No. Good! Now we must determine whether the subexpression is a formula.
- Is it an atomic formula? No.
- Is it of the form $\sim\mathbf{P}$? No.
- Is it of the form $(\mathbf{P}\ \&\ \mathbf{Q})$, $(\mathbf{P} \vee \mathbf{Q})$, $(\mathbf{P} \supset \mathbf{Q})$, or $(\mathbf{P} \equiv \mathbf{Q})$? Yes! So it is a formula if **P** and **Q** are formulas. **P** is Ax and **Q** is $(Bx \vee Cx)$
- Ax is an atomic formula. Let us now check $(Bx \vee Cx)$:
- It is not of form 1.
- It is not of form 2.
- It is of form 3.
- Sub-expressions: Bx, Cx are atomic formulas. We are done!

2.7 Sentences

$$\textbf{Expression} \\ \Downarrow \\ \textbf{Formula} \\ \Downarrow \\ \textbf{Sentence}$$

Sentence: special type of formula is which all the variables are **bound**.

2.7.1 Bound vs. free variables

A variable **x** occurring in a formula **P** can be either bound or free.

Bound variable: **x** occurs within the scope of an **x**-quantifier.

Free variable: not bound.

Example: $(\exists y)(Gxy \ \& \ Fa)$

- y is bound. Both occurrences are in the scope of a y-quantifier, $(\exists y)$.
- x is free. Not in the scope of an x-quantifier.
- a? neither bound nor free (a is not a variable).

2.7.2 Open sentence:

An open sentence is any formula that is **not** a sentence.

2.8 Substitution Instances

Given a sentence **P**,

We call **P(a/x)** a substititition instance of **P**.

It is like **P** except that it replaces every occurrence of **x** with **a**.

We read "(a/x)" as "a for x"

We call **a** the **instantiating constant.**

2.8.1 Procedure for producing a substitution instance

1. Drop the **initial** quantifier.
2. Replace **freed** variables with **the** desired constant (the same one each time).

Example: $(\forall x)(Fx \supset Gx)(a/x)$

1. Drop initial quantifier: $(Fx \supset Gx)$
2. Substitute a for x: $(Fa \supset Ga)$

Only quantified sentences can have substitution instances.

- E.g., $\sim (\forall x)Fx$ is not a quantified sentence.

Only the first quantifier can be dropped.

- E.g., $(\forall x)(\forall y)Fxy \Rightarrow (\forall x)Fxa$. ***NO!***

Example: $(\exists y)(\forall z)(\forall x)[Uy \,\&\, (Lxz \supset Lxy)](c/y)$

- Drop initial quantifier: $(\forall z)(\forall x)[Uy \,\&\, (Lxz \supset Lxy)]$
- substitute c for y: $(\forall z)(\forall x)[Uc \,\&\, (Lxz \supset Lxc)]$

3 A,E,I,O Sentences

3.1 A-Sentences

Suppose we want to symbolize: "All cows have udders"

One way:
UD: Cows
Ux: x has an udder.

$\Rightarrow (\forall x)Ux$

But suppose we wanted to talk about living things in general instead of just cows? This would be useful, say, if we wanted to symbolize a conversation with someone about the animals on a farm. But in that case, $(\forall x)Ux$ says that all living creatures have udders, and this is not what our original sentence says.

So let us re-write our symbolization key:
UD: Living creatures
Cx: x is a cow.
Ux: x has an udder.

Now let us try to symbolize "all cows have udders" as follows:

$(\forall y)Cy$ & $(\forall y)Uy$

What does this say?

"for every y in the UD, y is a cow, and for every y in the UD, y has an udder." In other words, "everything is a cow and everything has an udder." But this certainly does not say the same thing as "all cows have udders"

Let us begin again.

Suppose the statement is true. Now pick one thing (anything) from the UD. Suppose it is a cow. What can we say about it? That it has an udder!

So if we arbitrarily pick something from the UD, then **If it is a cow, then it will have an udder**

Or in symbols: $(\forall x)(Cx \supset Ux)$

Similarly, let us add to the UD:
Bx: x is a bird
Ox: x is an ostrich
Ax: x lives in Australia
Lxy: x loves y
Mx: x is a man.
Rx: x is from Mars.

j: Janet.

Here are some symbolizations with this UD and symbolization key:

- "All birds love Janet" $(\forall x)(Bx \supset Lxj)$
- "Every Ostrich lives in Australia" $(\forall x)(Ox \supset Ax)$
- "Men are from Mars." $(\forall x)(Mx \supset Rx)$

In general, statements of the form "All Ps are Qs" are called **A-Sentences**.

3.2 E-Sentences

Let us add to the UD:
Px: x is a pig.
Fx: x can fly.

Consider the sentence "No pigs can fly."

Suppose we pick an arbitrary living thing from the UD. What can we say about it if it is a pig?

- That it can't fly.

So ... "for everything in the UD, if it is a pig, then it is not the case that it can fly".

In symbols, $(\forall x)(Px \supset\sim Fx)$

Sentences of the form: "No Ps are Qs" are called **E-Sentences**.

3.3 I-Sentences

Let us add to the UD:

Sx: x is a swan

Kx: x is black

Consider: “Some swans are black”

Remember, “some” is usually symbolized with $\exists$ which is read as “There is at least one”

So let’s begin: “There is at least one member of the UD, x, such that ...”

Such that what? What can we say about this thing?

1. it is a swan
2. it is black

$\Rightarrow (\exists x)(Sx \;\&\; Kx)$

There are other ways to say this:

- **There are** black swans
- Black swans **exist**
- etc.

Question: Why can’t we symbolize this sentence this way: $(\exists x)(Sx \supset Kx)$?

Answer: The original sentence is asserting existence. It is asserting that something having the specified property exists. But this symbolization will be true even if all swans have gone extinct. For anything in the UD that you pick, the antecedent will be false of it, since nothing is a swan. Therefore the whole sentence will true of it.

Sentences of the form “Some As are Bs” are called **I-Sentences**.

3.4 O-Sentences

Consider: “Some birds cannot fly”

This is similar to the I-Sentence form.

"There is at least one x in the UD such that ..."

- it is a bird
- it cannot fly

$\Rightarrow (\exists x)(Bx \ \& \ \sim Fx)$

Sentences of the form "Some As are not Bs" are called **O-sentences**.

3.5 Square of Opposition

$$\mathbf{A}:(\forall \mathbf{x})(\mathbf{P} \supset \mathbf{Q}) \qquad \mathbf{E}:(\forall \mathbf{x})(\mathbf{P} \supset \sim \mathbf{Q})$$

$$\mathbf{I}:(\exists \mathbf{x})(\mathbf{P} \ \& \ \mathbf{Q}) \qquad \mathbf{O}:(\exists \mathbf{x})(\mathbf{P} \ \& \ \sim \mathbf{Q})$$

Contradictories:

A is the contradictory of O.

In other words A is the opposite, i.e, the negation of O.

- So $(\forall x)(Px \supset Qx)$ is equivalent to $\sim (\exists x)(Px \ \& \ \sim Qx)$
- And $(\exists x)(Px \ \& \ \sim Qx)$ is equivalent to $\sim (\forall x)(Px \supset Qx)$

E is the contradictory of I

- So $(\forall x)(Px \supset \sim Qx)$ is equivalent to $\sim (\exists x)(Px \ \& \ Qx)$
- And $(\exists x)(Px \ \& \ Qx)$ is equivalent to $\sim (\forall x)(Px \supset \sim Qx)$

Question: Does an A-Sentence imply an I-Sentence?

No. Why not?

Because it could be the case that no Ps exist.

Suppose that Dx stands for: x is a dodo-bird.

Then the A-sentence is true.

But the I-Sentence is false because no such creature is in the UD.

Similarly for E and O sentences.

Note also:

A-Sentence and E-Sentences are compatible

This is true when nothing satisfies the antecedent predicate.

“All Dodos are ugly”, “All Dodos are not ugly” are both trivially true when Dodos do not exist.

I and O sentences are also compatible since they also can both be true at the same time (note that they can also both be *false* at the same time).

3.6 Shortened forms of A,E,I,O

We use shortened forms when the things we are referring to constitute the entire domain.

UD: People

Hx: x is happy

A: Everyone is happy $(\forall x)Hx$

E: No one is happy $(\forall x) \sim Hx$

I: Some people are happy $(\exists x)Hx$

O: Some people are not happy $(\exists x) \sim Hx$

3.7 Examples

UD: Living things

Bx: x is a bear
Dx: x is a duck
Hx: x is hungry
Gx: x is a goose
Lx: x likes fish
Px: x is a pig

1) "All hungry bears like fish"

⇒ "for every x in the UD, if x is a <what?> then it likes fish"

What does x have to be? A hungry bear!

How do you say that x is a hungry bear? $Hx \,\&\, Bx$

⇒ $(\forall x)[(Hx \,\&\, Bx) \supset Lx]$

This is an A-sentence. What is its contradictory? $\sim (\exists x)[(Bx \,\&\, Hx) \,\&\, \sim Lx]$

2) "Ducks and geese can fly, but pigs can't"

This is a conjunction of two A-Sentences and one E-Sentence:

$$[(\forall x)(Dx \supset Fx) \,\&\, (\forall x)(Gx \supset Fx)] \,\&\, (\forall x)(Px \supset\sim Fx)$$

3) UD: People
Lx: x is a logician
Ixy: x has been influenced by y
Gxy: x is greater than y.
a: Alfred Tarski
b: Bertrand Russell

"No logician is greater than Bertrand Russell"

Pick an x in the UD. If it is a logician, then what can we say?

That it is not greater than Bertrand Russell.

⇒ $(\forall x)(Lx \supset\sim Gxb)$

This is an E-Sentence. What is its contradictory?

$\sim (\exists x)(Lx \,\&\, Gxb)$

4) “Most logicians have been influenced by either Alfred Tarski or Bertrand Russell”

Note that for us, “most” means the same as “some”.

So there is at least one logician who has what?

Been influenced by Russell or been influenced by Tarski.

i.e., $(\exists x)[Lx \mathbin{\&} (Ixa \vee Ixb)]$

This is an I-sentence. What is its contradictory?

$\sim (\forall x)[Lx \supset \sim (Ixa \vee Ixb)]$

4 Symbolization Techniques

Recall that in the A-Sentence-schema: $(\forall x)(\mathbf{P} \supset \mathbf{Q})$,

$\mathbf{P}$ and $\mathbf{Q}$ are metavariables and can be replaced with more complex expressions.

Sample A-Sentences:

- $(\forall x)[(Ax \vee Bx) \supset (Ax \supset Cx)]$
- $(\forall x)([Ax \mathbin{\&} (Bx \supset Cx)] \supset [Dx \equiv (Cx \mathbin{\&} Ex)])$

Similarly for E, I, and O.

4.1 Examples:

UD: People
Lxy: x loves y
a: Frege
b: Russell
c: Wittgenstein

“Everyone who loves Frege loves Russell or Wittgenstein.”

“Everyone” means every person, so:

“For every x in the UD, x is such that ...”

Pick any x in the UD, what can we say?

If it loves Frege, then it also loves Russell or Wittgenstein.

So: “For every x in the UD, x is such that if **it** likes Frege, then **it** likes either Russell or Wittgenstein.”

In symbols: $(\forall x)[Lxa \supset (Lxb \vee Lxc)]$

This is an **A-Sentence**.

- **P**: Lxa
- **Q**: $Lxb \vee Lxc$

Let us add some more predicates:
Lx: x is a logician
Fx: x is famous
Wx: x is wealthy
Gx: x is good-looking

“Many famous logicians are neither good-looking nor wealthy”

So there is at least one person who is what?

Both famous and a logician.

$(\exists x)[(Fx \ \& \ Lx)$...

What about this person?

They are not wealthy, and not good-looking.

$(\exists x)[(Fx \ \& \ Lx) \ \& \ \sim (Wx \vee Gx)]$

This is an O-sentence. What is it equivalent to?

- Negation of an A-sentence:
- $\sim (\forall x)[(Fx \ \& \ Lx) \supset (Wx \vee Gx)]$

Let us add another predicate:

Ixy: x has been influenced by y.

"Frege has influenced everyone that has been influenced by Russell or Wittgenstein"

Here, "Everyone that has been influenced by Russell or Wittgenstein" gives us a clue that we will likely need a universal quantifier.

So pick something in the UD. Let's say that it has been influenced by either Russell or Wittgenstein. What can we say about it?

That it has been influenced by Frege!

"For every x in the UD, if it has been influenced by either Russell or Wittgenstein, then it has been influenced by Frege."

In symbols: $(\forall x)[(Ixb \vee Ixc) \supset Ixa]$

This is an **A-Sentence**. **P**: $Ixb \vee Ixc$, **Q**: Ixa.

Let us add more predicates:

Ax: x is an actor

Rx: x is a rock-star

'Actors and Rock-stars are famous, but Logicians are not."

So, "For every x in the UD, if it is ..."

an actor, what can we say? It is famous

a rock-star? Famous

a logician? Not famous

In symbols: $(\forall x)(Ax \supset Fx)$ & $(\forall x)(Rx \supset Fx)$ & $(\forall x)(Lx \; \& \; \sim Fx)$

This is a conjunction of two A-sentences and one O-sentence.

Consider the following sentence:

"Anyone who has been influenced by Wittgenstein has been influenced by Russell."

$\Rightarrow$ $(\forall x)(Ixc \supset Ixb)$ Here "anyone" is interpreted to mean the same thing as"every", "all", etc.

But how should we interpret "anyone" in this sentence?

"If anyone has been influenced by Frege, Russell has"

The sentence is in fact better interpreted as:

"If there is at least one thing in the UD that has been influenced by Frege, then Russell has been influenced by Frege."

We need to use an *existential* quantifier.

$\Rightarrow$ $(\exists x)Ixa \supset Iba$

How do we know when to use universal or existential when translating "anyone", "someone", etc.?

In some cases, we can use the presence of **pronominal cross-reference** to help us decide.

- "pronomial" $\Rightarrow$ **pronoun**, i.e., he, she, it, etc.

"If **Socrates** eats hemlock, **he** will die."

- the reference of the pronoun "he" is the noun "Socrates"
- the pronoun in the consequent references something in the antencedent. This is called "cross-reference"

Importantly for our purposes, pronominal cross-reference can also happen between a pronoun and a **quantity term** ('a', 'any', 'some', 'all', 'every', etc.).

e.g.: "If anyone breaks the rules, she will be punished."

- "she" cross-references the quantity-term "anyone"

e.g. "If anyone breaks the rules, Michael will be punished."

- Here there is no pronominal cross-reference.
- The two clauses of the sentences are separable.

Pronominal Cross-Reference (PCR) Rule:

- In an English conditional...
- If a quantity term is used in the antecedent,
- And if, in the consequent, there is pronominal cross-reference to the quantity term,
- Then use a universal quantifier.

e.g., "If someone breaks the rules, she will be punished."

- $(\forall x)(Bx \supset Px)$
- Note the use of a universal quantifier in spite of the occurrence of "someone". Normally "someone" calls for the use of an existential quantifier, but not when there is pronominal cross-reference!

e.g., "If everyone breaks the rules, Michael will be punished."

- No pronominal cross reference here.
- This means we will have two separate clauses.
- Does this mean we have to use an existential quantifier?
- **NO!**

The rule says that **if there is** pronominal cross-reference, then we need to use a universal quantifier.

It says nothing about what to do when there isn't any pronominal cross-reference.

When there isn't pronominal cross-reference, we might have to use a universal quantifier or we might have to use an existential quantifier.

In this case, "everyone" clearly indicates the use of a universal quantifier: $(\forall x)Bx \supset Pm$

e.g., "Every good boy deserves fudge."

- There is no obvious pronominal cross-reference here.
- But this seems like an A-Sentence.
- What should we do?

We must attempt to paraphrase the sentence into a conditional involving pronominal cross-reference. If we can do this, then we know we must use a universal quantifier.

"Every good boy deserves fudge." can be paraphrased as follows:

Imagine picking something from the UD. If it is a boy, then what can we say? That it deserves fudge.

So we can paraphrase the sentence as follows:

⇒ "If **any** boy is good, then **he** deserves fudge"
UD: boys
Gx: x is good
Fx: x deserves fudge

$(\forall x)(Gx \supset Fx)$

4.2 The Principle of Charity

Consider:

UD: people
Px: x is a politician
Lx: x is a liberal
Cx: x is a conservative

"Politicians are both Liberal and Conservative"

When no quantity term ('all', 'some', etc.) is specified, we normally assume 'all' if there are no other clues.

So this sentence is straightforwardly symbolized as ⇒ $(\forall x)(Px \supset (Lx \ \&\ Cx))$, right?

No. This doesn't make sense in context. Someone can't be both a liberal and a conservative.

Rule of thumb: use the **principle of charity**: try to give the interpretation that makes the most sense in the given context (as long as it does not contradict anything that has been said).

This person is obviously trying to say: "some politicians are liberals and some politicians are conservatives"

$(\exists x)(Px \ \&\ Lx) \ \&\ (\exists x)(Px \ \&\ Cx)$

4.3 More examples ...

UD: politicians
Lx: x is a liberal
Cx: x is a conservative
Dx: x is a new democrat
Rx: x is reasonable
Ux: x is under 70
Bxy: x is a better debater than y
Axy: x agrees with y
a: Stephen Harper
b: Michael Ignatieff
c: Jack Layton

1) “There are reasonable liberals but no reasonable new democrats”

- “but” splits the sentence into two clauses. The sentence is a truth-functional compound.
- First clause: ““there is at least one reasonable liberal” ⇒ $(\exists x)(Rx \ \& \ Lx)$
- Second Clause: “no reasonable new democrats” ⇒ $\sim (\exists x)(Rx \ \& \ Dx)$.
- All together: $(\exists x)(Rx \ \& \ Lx) \ \& \ \sim (\exists x)(Rx \ \& \ Dx)$

2) “Stephen Harper is a conservative and he is a better debater than any liberal under 70.”

Two clauses:

1. “Stephen Harper is a conservative”
2. “Stephen Harper is a better debater than any liberal under 70.”

Clause 1:

- Ca

Clause 2:

- Paraphrase: “If **a** liberal is under 70, then Stephen Harper is a better debater than **her**.”
- Pronominal cross-reference ⇒ Universal quantifier.
- All together: Ca & $(\forall x)[(Lx \ \& \ Ux) \supset Bax]$

3) “Every new democrat is a better debater than Stephen Harper, but they are not all better debaters than Michael Ignatieff” Two clauses again, separated by “but”

First clause: “Every new democrat is a better debator than Stephen Harper”

Try to rephrase this with pronominal cross-reference: “If **any** politician is new democrat, then **he** is a better debater than Stephen Harper” ⇒ Success.

Second clause: “they are not all all better debators than Michael Ignatieff”

Here, “they” refers to New Democrats in the first clause (note “new democrats” is a noun, not a quantity term so the pronominal cross-reference can be ignored).

$\Rightarrow (\forall x)(Dx \supset Bxa)\ \&\ \sim (\forall x)(Dx \supset Bxb)$

4) “Jack Layton disagrees with both reasonable and unreasonable conservatives alike”

There are two claims being made:

1. “Jack Layton disagrees with reasonable conservatives”
2. “Jack Layton disagrees with unreasonable conservatives”

$\Rightarrow (\forall x)[(Rx\ \&\ Cx) \supset\sim Acx]\ \&\ (\forall x)[(\sim Rx\ \&\ Cx) \supset\sim Acx]$

We can also shorten it like so: $(\forall x)([(Rx\ \&\ Cx) \vee (\sim Rx\ \&\ Cx)] \supset\sim Acx)$

4.4 ‘Stacked-up’ Adjectives

“Any reasonable liberal under 70 agrees with Jack Layton.”

- To symbolize “any reasonable liberal under 70”, we can just conjoin the appropriate predicates:
- $(\forall x)([(Rx\ \&\ Lx)\ \&\ Ux] \supset Axc)$

In general, try to make your predicates as fine grained as possible.

But look out for the following:

4.4.1 Adjective modifiers

A “large mouse” is not something that is both large and a mouse (it is large *for* a mouse).

A “second-rate mathematician” is not both second-rate and a mathematician. She is second-rate *for* a mathematician.

In cases like these, we must use just one predicate to capture the whole description:

Lx: x is a large mouse.

Mx: x is a mouse.

4.4.2 Entities that do not exist in the UD

Consider: “1 is the successor of 0”

Can we translate it as:

Sab?

where:
UD: Positive Integers
Sxy: x is the successor of y
a: 1
b: 0

No. 0 does not exist in this UD (0 is not a positive integer) so it cannot be represented by an individual constant.

How about:

$(\exists x)(Zx \: \& \: Sax)$?

where:
UD: Positive Integers
Sxy: x is greater than y
Zx: x is 0
a: 1

No. This says that there exists an x **in the UD** that is 0. This is false.

How about $(\forall x)(Zx \supset Sax)$

No. This would be true no matter what **a** was. It would be true if **a** was 2000, for instance. This is because nothing in the UD is 0, so no matter what x you pick, it will be true that **if** it is 0, 2000 is its successor. So the antecedent is trivially satisfied and the sentence is true for all x.

Solution:

We have no choice but to use a one-place predicate when we want to specify a relation with something that is not in the UD.

Zx: x is the successor of 0.

Note that the same problem occurs for the following case:

- "Jim is taller than Socrates" where the UD is living people.
- The problem occurs because Socrates is dead. He doesn't exist in **this UD**.
- Thus we have no choice but to use the one-place predicate Sx ("x is taller than Socrates") and then translate the sentence as Sj.
- However the problem would not be present if we used a UD of, e.g., "all people who have ever lived". Socrates *does* exist in this second UD!

4.4.3 Action Predicates

- "Looking", "searching", "sailing to", etc.

Consider:

"John is looking for a prime number greater than $7,000$"

Can we symbolize this like so?

UD: everything
Px: x is a prime number
Sxy: x is greater than y

Lxy: x is looking for y
j: John
s: 7,000

$(\exists x)[(Px \;\&\; Sxs) \;\&\; Ljx]$

No.

- Is it true that he is looking specifically for 7,001? No.
- Is it true that he is looking specifically for 7,013? No.
- Is it true that he is looking specifically for 7,019? No.
- Is it true that he is looking specifically for 7,001, 7,027, and 7,079? No.
- Is it true that he is looking specifically for 7,001 and 49,957? No.

None of these are true. It is not the case that John is looking for a particular number or numbers. He would be happy with any old number greater than 7,000. So this translation will not do either.

How about the following symbolization?

$(\forall x)[(Px \;\&\; Sxs) \supset Ljx]$

No. This says that John is looking for *all* primes greater than $7,000$. This is not what our original sentence said.

Again, there is no choice but to use a single-place predicate:

- Lj
- where Lx: x is looking for a prime greater than $7,000$

On the other hand, if the sentence were:

"John is looking for the **least** prime greater than $7,000$"

then we could say:

$(\exists x)[(Px \;\&\; Sxs) \;\&\; Ljx]$

where now **Sxy:** x is the **least** number greater than y.

We can do this since it is true that there is a particular number that John is looking for (there is only one least prime greater than 7,000).

Note that the use of the definite article "the" gives us a clue that there is only one thing in this case.

Rule:

if the object of the action verb (searching, hunting, etc.) is not a particular thing, then use a 1-place predicate.

5 Quantifiers with Overlapping Scope

5.1 The basics

Remember, when the scopes of two (or more) quantifiers overlap, we must use a different variable for each quantifier:

$(\forall x)Ax \vee (\forall x)Bx \Rightarrow$ OK!

$(\forall x)(\forall x)Rxx \Rightarrow$ *No!*

$(\forall x)(\forall y)Ryx \Rightarrow$ *OK!*

$(\exists x)[Ax \equiv (\forall x)(Bx \supset Cx)]$ *No!*

$(\exists x)[Ax \equiv (\forall y)(Bx \supset Cy)]$ *OK!*

Quantifiers with overlapping scope express sentences that are more complex than the simple sentences we have been dealing with so far.

The simplest type of sentence containing quantifiers with overlapping scope is when when one quantifier immediately follows another.

There are 4 possibilities, which we usually translate as follows:

$(\forall x)(\forall y)$: "Every x in the UD is such that for every y in the UD ..."

$(\forall x)(\exists y)$: "For every x in the UD there is at least one y in the UD such that ..."

$(\exists x)(\forall y)$: "There is at least one x in the UD such that for every y in the UD ..."

$(\exists x)(\exists y)$: "There is at least one x and at least one y in the UD such that ..." **or** "There is a pair x and y in the UD such that ..."

5.2 Examples

Consider the following symbolization key:

UD: People
Rxy: x respects y

Example 1: $(\forall x)(\forall y)Rxy$

Quasi-english: ⇒ "Every x in the UD is such that for every y in the UD, x respects y."

Better english: ⇒ "Everyone is such that they respect everyone in the UD"

Fluent english: ⇒ "Everyone respects everyone"

Example 2: $(\forall x)(\exists y)Rxy$

Quasi-english: ⇒ "For every x in the UD there is at least one y such that x respects y."

Better english: ⇒ "For everyone, there is at least one person that they respect."

Fluent english: ⇒ “Everyone respects someone”

Example 3: $(\exists x)(\forall y)Rxy$

Quasi-english: ⇒ “There is at least one x in the UD such that for every y in the UD, x respects y.”

Better english: ⇒ “At least one person respects everyone”

Fluent english: ⇒ “Someone respects everyone”

Example 4: $(\exists x)(\exists y)Rxy$

Quasi-english: ⇒ “There is at least one x and at least one y in the UD such that x respects y”

Better english ⇒ “There is at least one person that respects at least one person.”

Fluent english: ⇒ “Someone respects someone.”

Let’s add some more predicates:
Px: x is a philosopher.
Lx: x is a logician.

Example 5: “Every philosopher respects every logician”

The sentence seems like a universally quantified sentence. So let us try our usual procedure. Pick something in the UD. Suppose it is a philosopher. Can we say anything about him?

Yes, we can say that he *respects every logician.* Now we will try to symbolize that.

Imagine that after we pick the philosopher, we pick another thing from the UD.[2] Suppose that that other thing is a logician. Then in that case, we can say that the philosopher respects it. So if it is a logician, the philosopher will respect it.

Here is a paraphrase of our full sentence: "If *anyone* is a philosopher, then if *anyone* logician, then *he* will respect *her.*"

(Note that there is pronominal cross-reference (PCR) here for both quantifiers.)

In symbols: $\Rightarrow (\forall x)(Px \supset (\forall y)(Ly \supset Rxy))$

Example 6: "Some philosophers do not respect logicians."

Rough paraphrase: "*Some* philosophers are such that *they* do not respect logicians."

There is cross-reference from a pronoun (they) to a quantity term (some) here, but this sentence is not in the form of a conditional and we cannot put it in that form without changing its meaning.

So let us begin: "There is at least one philosopher such that ..."

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[2] Actually it doesn't need to be another thing. We could conceivably pick the same person again. The UD is like a deck of cards. Imagine picking one card, taking note of it, and then putting it back in the deck. If you now pick another card, then of course it is possible to pick the same card again!

What about these philosophers? They do not respect (any) logicians. So if we were to pick a logician from the UD, then we know that those philosophers would not respect it.

In quasi-english: “There is at least one person who is a philosopher who is such that if *anyone* is a logician, then the philosopher does not respect *him*.”

Note that there is PCR here, but only for the part of the sentence that begins: “if anyone”

In symbols: $\Rightarrow (\exists x)(Px \ \& \ (\forall y)(Ly \supset \sim Rxy))$

Example 7: “Every logician is not respected by some philosophers”

If we pick something from the UD and it is a logician, then what can we say?

That there is at least one philosopher that does not respect it.

“If *anyone* is a logician, then there is at least one philosopher that does not respect *her*.”

There is PCR here from her to anyone.

In symbols: $\Rightarrow (\forall x)(Lx \supset (\exists y)(Py \ \& \ \sim Ryx))$

Example 8: “There are some philosophers who are not respected by some logicians”

Quasi-english: “There is at least one thing in the UD such that it is a philosopher and such that there is at least one thing in the UD that is a logician who does not respect her.”

No PCR here (there are cross-referencing pronouns but the sentence is not in the form of a material conditional).

In symbols: $\Rightarrow (\exists x)(Px \ \& \ (\exists y)(Ly \ \& \ \sim Ryx))$

Let's change our symbolization key:

UD: positive integers
Lxy: x is larger than y

Example 9: "There is a smallest positive integer."

A first attempt: ⇒ "There is an x in the UD such that x is the smallest positive integer."

What does it mean to be the smallest positive integer? It means that no positive integer is larger than it.

So if we pick something from the UD, what can we say? That it will not be larger than x.

In quasi-english: ⇒ "There is an x in the UD such that for every y in the UD, x is not larger than y."

In symbols: $\Rightarrow (\exists x)(\forall y)(\sim Lxy)$

Let's add some new predicates:

Sxyz: x is the sum of y and z
Ex: x is even

Ox: x is odd

Example 10: "The sum of two odd numbers is even"

Clearly this claim is meant to be universal. It says that any two odd numbers, when added together, give an even number.

So suppose you arbitrarily pick something from the UD and it is odd. Then what can you say?

Well you can say that if you pick again and you find something that is also odd, then the sum of these two things is even.

In symbols: $\Rightarrow (\forall x)(Ox \supset (\forall y)(Oy \supset ...$

We are almost done. We now need to symbolize that the sum of x and y is even.

Well suppose you pick something and it happens to be *the sum of x and y.* What can you say about *the sum of x and y*?

That it is even. So ...

In symbols: $\Rightarrow (\forall x)(Ox \supset (\forall y)[Oy \supset (\forall z)(Szxy \supset Ez)])$

Example 11: Goldbach's Conjecture (1742)

Goldbach's conjecture is one of the oldest unsolved problems in mathematics. It reads:

"Every even integer greater than 2 can be expressed as the sum of two primes."

Let us use the following symbolization key:

Gxy: x is greater than y
Px: x is prime
a: 2

Ok. Suppose you arbitrarily pick something from the UD and it is even and greater than 2. What can you say about it?

That it can be expressed as the sum of two primes.

How do we symbolize: “can be expressed as the sum of two primes.” ? What is this saying?

It is saying that there is a pair of prime numbers such that their sum is x.

In symbols: ⇒ $(\forall x)((Ex \,\&\, Gxa) \supset (\exists y)(\exists z)((Py \,\&\, Pz) \,\&\, Sxyz))$

5.3 Equivalences

Usually, if a quantifier ranges over just a part of a sentence, it can be extended to range over the whole sentence without changing the meaning of the sentence.

Examples:

- $\mathbf{P} \supset (\exists x)Ax$ and $(\exists x)(\mathbf{P} \supset Ax)$
- $\mathbf{P} \supset (\forall x)Ax$ and $(\forall x)(\mathbf{P} \supset Ax)$
- $(\exists x)Ax \vee \mathbf{P}$ and $(\exists x)(Ax \vee \mathbf{P})$
- $(\forall x)Ax \ \& \ \mathbf{P}$ and $(\forall x)(Ax \ \& \ \mathbf{P})$
- $\mathbf{P} \ \& \ (\forall x)Ax$ and $(\forall x)(\mathbf{P} \ \& \ Ax)$
- and others ...[3]

Exceptions:

1. You cannot do this at all with **Biconditionals**.
    - $(\exists x)Ax \equiv \mathbf{P}$ is ***NOT*** equivalent to $(\exists x)(Ax \equiv \mathbf{P})$
    - $\mathbf{P} \equiv (\exists x)Ax$ is ***NOT*** equivalent to $(\exists x)(\mathbf{P} \equiv Ax)$
    - (similarly with a universal quantifier)
2. For a **material conditional**, you cannot do this if:
    (a) The quantifier comes before the **antecedent**
    (b) And the **consequent** is brought within or taken out of the scope of the quantifier.
    - $(\exists x)Ax \supset \mathbf{P}$ is ***NOT*** equivalent to $(\exists x)(Ax \supset \mathbf{P})$
    - (similarly with a universal quantifier)

However, for material conditionals, it is still possible to extend or shrink the scope of the quantifier to include or exclude the consequent as long as you *change the quantifier* when you do so (i.e. from existential to universal or vice versa).

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[3] **Note**: We assume that x does not occur in **P**. If x does occur in **P** and if you are **extending** the scope of an quantifer, then you will need to change one of the variables first. E.g. $(\forall x)Ax \ \& \ (\forall x)Bx$ becomes $(\forall x)(Ax \ \& \ (\forall y)By)$.

Valid Equivalences:

- $(\exists x)Ax \supset \mathbf{P} \Leftrightarrow (\forall x)(Ax \supset \mathbf{P})$
- $(\forall x)Ax \supset \mathbf{P} \Leftrightarrow (\exists x)(Ax \supset \mathbf{P})$
- $(\exists x)(Ax \supset \mathbf{P}) \Leftrightarrow (\forall x)Ax \supset \mathbf{P}$
- $(\forall x)(Ax \supset \mathbf{P}) \Leftrightarrow (\exists x)Ax \supset \mathbf{P}$

Example:
UD: Students in Poetry class
Tx: x passed the test.
a: Andrew

"If anyone passed the test, Andrew passed the test."

One way to symbolize this sentence is as follows:

$(\exists x)Tx \supset Ta$

Which means: $\Rightarrow$ "If at least one x in the UD passed the test, then Andrew passed the test."

This is not equivalent to:

$(\exists x)(Tx \supset Ta)$

To see why, suppose that Andrew failed but that someone else (say Bob) in the class passed.

In this case, the antecedent $(\exists x)Tx$ is true, since it is the case that at least one person passed the test (Bob did).

But the consequent, Ta is false, because it is not the case that Andrew passed.

So the material conditional, $(\exists x)Tx \supset Ta$ is *false* since the antecedent is true but the consequent is false.

But what about the statement $(\exists x)(Tx \supset Ta)$? In fact this statement is *true.*

Why? Because there is someone in the UD that this is true of: *Andrew.*

The sentence is saying this: "There is at least one x in the UD such that *if that x passed the test, then Andrew passed the test*"

Andrew is a member of the UD. So let's see if he satisfies this condition. Is it true that: "If Andrew passed the test, then Andrew passed the test" ?

Let's look at the antecedent. It is false, since Andrew failed. What do we say about a material conditional whose antecedent is false? We say that the material conditional is true.

So it is true of Andrew, that *if he passed, then he passed.*

But now it's clear that $(\exists x)Tx \supset Ta$ and $(\exists x)(Tx \supset Ta)$ have different truth conditions and thus do not say the same thing. They are therefore *not* equivalent.

On the other hand, $(\exists x)Tx \supset Ta$ is equivalent to $(\forall x)(Tx \supset Ta)$

To see why:

1) Imagine that $(\exists x)Tx \supset Ta$ is true.

So this means that if there is someone in the UD that passed, then Andrew passed.

So pick someone, *anyone*, from the UD.

Imagine that that person passed.

Well if *that person* passed, then *someone* passed.

But we just said that if *someone* in the UD passed, then Andrew passed.

So this means that Andrew must have passed.

So, to sum up: no matter who we pick from the UD: *if* they passed, *then* Andrew passed.

In quasi-english, for every x in the UD, if they passed, Andrew passed:

Or, in symbols: $(\forall x)(Tx \supset Ta)$

2) Ok, now imagine that $(\forall x)(Tx \supset Ta)$ is true.

This means that for every x in the UD, if they passed, Andrew passed.

So let's suppose that at least one person passed.

Well if that person passed, then we know that Andrew must have passed (since for every x in the UD, if they passed, Andrew passed).

So if at least one person passed, Andrew passed.

Or in symbols: $(\exists x)Tx \supset Ta$

We derived $(\exists x)Tx \supset Ta$ from $(\forall x)(Tx \supset Ta)$ and vice versa. So the two sentences are **equivalent.**

Similarly for:

$(\forall x)Ax \supset \mathbf{P}$ and $(\exists x)(Ax \supset \mathbf{P})$

5.4 More examples:

Example 1:
UD: people
Wx: x is a woman
Mx: x is a man
Axy: x admires y
Rxy: x respects y
Lxy: x loves y
m: Martha

"Martha loves men who admire self-respecting women."

One useful strategy is to, as a first step, try and paraphrase it in the general form of one of the A, E, I, O sentences (or their negations). Recall that the english forms of these are:

A: "All Ps are Qs"

E: "No Ps are Qs"

I: "Some Ps are Qs"

O: “Some Ps are not Qs”

So let us return to the sentence about Martha. Clearly, the intention of this sentence is universal. The sentence is trying to say something about all men. No negations occur in this sentence, so let’s try to paraphrase it as an A-sentence. In fact, we can do this as follows:

“All men who admire self-respecting women are loved by Martha”

Ok. So pick something from the UD and imagine that it is ...

Hmmm... it seems like there is some trouble even at this early stage. What must the thing that we pick from the UD be like? Well, it must be:

1. a man
2. who admires self-respecting women.

The first condition is easy to symbolize: Mx.

The second condition says that x must admire self-respecting women. This is a little complex so we need to think about how to symbolize it. It is clearly saying that this man must admire *all* self-respecting women.

So let’s pick another thing from the UD. Suppose it is a self-respecting woman, then what can we say? Well we can say that the man must admire her.

So we can symbolize the second condition as: $(\forall y)((Ryy \mathbin{\&} Wy) \supset Axy)$

We can conjoin the first and second conditions as follows:

$(\forall x)[(Mx \mathbin{\&} (\forall y)((Ryy \mathbin{\&} Wy) \supset Axy))...$

So, finally, if x is a man who admires self-respecting women, what can we say about him?

That Martha loves him.

Let's put it all together:

$(\forall x)[(Mx \mathbin{\&} (\forall y)((Ryy \mathbin{\&} Wy) \supset Axy)) \supset Lmx]$

Example 2:
UD: everything
Lx: x is a logician
Ox: x is a person
Px: x is a philosopher
Sx: x is a second-rate philosopher
Gx: x is a good philosopher
Rxy: x reads y
Wxy: x is the work of y
Ixy: x ignores y
Uxy: x understands y
Exy: x envies y

"Many people read only the work of second-rate philosophers"

Remember that "Many" means the same as "some" or "at least one" for our purposes.

This looks like an I-sentence. "Some Ps are Qs"

So there are some people in the UD, $(\exists x)Ox$ & ...

and what can we say about these people?

They will read something only if it is the work of a philosopher who is second-rate.

Let us begin by starting in quasi-english: "There is at least one person, x, in the UD such that for every thing, y, in the UD, x will read y only if ... *what?*"

Only if "there is some z, that y is the work of, that is a second-rate philosopher"

$(\exists x)(Ox \ \& \ (\forall y)(Rxy \supset (\exists z)(Wyz \ \& \ Sz)))$

Example 3: "Good philosophers who understand any of the work of any logician are envied by other philosophers"

This looks like an A-Sentence.

It is saying something about good philosophers. I.e., if they are good philosophers, then something will follow. I.e.:

$(\forall x)Gx \supset ...$

Ok, well what about them?

It says that if they understand any of the work of any logician, then they are envied by other philosophers.

Let's focus on the first part: "if they understand any of the work of any logician"

We can change this into quasi-english as follows: “If there is some y that is the work of some z that is a logician, and if x can understand y, then ...

In symbols: $(\forall x)(Gx \supset [(\exists y)(\exists z)((Wyz \;\&\; Lz) \;\&\; Uxy) \supset ...$

Ok what can we say about such people?

That (all) other philosophers envy them - this last part is also in the form of an A-sentence.

$(\forall x)(Gx \supset [(\exists y)(\exists z)((Wyz \;\&\; Lz) \;\&\; Uxy) \supset (\forall w)(Pw \supset Ewx)])$

Example 5: “There are no good philosophers who ignore the work of logicians”

“There are no” - it looks like the negation of an I-sentence.

So ... it is not the case that there is anyone in the UD such that she is a good philosopher, and such that she ... what? *ignores the work of logicians.*

Let’s focus on that last part of the sentence. It says that anything that is the work of any logician is ignored by such people.

In symbols:

$\Rightarrow\sim (\exists x)[Gx \;\&\; (\forall y)(\forall z)[(Wyz \;\&\; Lz) \supset Ixy]$

6 Informal Semantics

6.1 Interpretation of PL

Given a sentence of PL, an interpretation of that sentence defines every *individual constant symbol* and every *predicate symbol* occurring in that sentence (we do not interpret variables).

Depending on the interpretation we give to a sentence, it may be true or false.

We represent an interpretation with a *symbolization key.*

For example, consider the sentence $(\exists x)(Ex \;\&\; Ox)$.

This sentence is *true* on this interpretation:

UD: people
Ex: x is English
Ox: x is old

But it is *false* on this one:

UD: positive integers
Ex: x is even
Ox: x is odd

6.1.1 Individual Constants

We interpret *constants* by assigning to them a member of the UD.

E.g.:

UD: people
a: Alice
b: Bob
c: Carol
etc.

(Note: remember that you *cannot* assign something to a constant that is *not* a member of the UD).

6.1.2 Predicates

We interpret predicates by assigning them an *extension*.

Extenstion of a predicate: the set of members of the UD that are picked out by the predicate.

E.g.

UD: Positive integers
Ex: {2, 4, 6, 8, 10, 12, 14, ... }
Ox: {1, 3, 5, 7, 9, 11, 13, ...}

Here, we have assigned the set {2, 4, 6, 8, 10, 12, 14, ... } to the predicate E and the set {1, 3, 5, 7, 9, 11, 13, ...} to the predicate O.

If we now interpret the constant a as:

a: 4

then we can say that the sentence Ea is true since a is in the extension of E. Oa is false, however, since a is not in the extension of O.

Similarly, $(Ea \supset Oa)$ is false since a is in the extension of E but not in the extension of O.

Normally we do not explicitly specify the extensions of our predicates.

Instead, we *describe* the members that comprise the extension (e.g., "Ex: x is even" or "Ox: x is odd"). We call this the *intension* of the predicate.

However, it is important to keep in mind that the proper interpretation of a predicate is in terms of its *extension*, not its *intension*.

Example:

UD: people
Px: x is the prime minister of Canada

- *extenstion of P:* {Stephen Harper}.

Kx: x is the king of Canada

- *extension of K:* $\varnothing$.[4]

Fx: x is a full-time faculty member in the Philosophy Department at UWO

- *extension of F*: {Lorne Falkenstein, Corey Dyck, Wayne Myrvold, John Bell, William Harper, Robert DiSalle, ... }.

Extensions of n-place predicates

Consider a UD of positive integers, and the 2-place predicate: *Gxy*: x is greater than y.

---

[4] $\varnothing$ signifies the *empty set*: the set containing no things.

The extension of this predicate is the set of *ordered pairs* of positive integers such that the first one is greater than the second:

$\{\langle 2, 1\rangle$, $\langle 3, 1\rangle$, $\langle 4, 1\rangle$, $\langle 7, 6\rangle$, $\langle 900, 17\rangle$, ...$\}$

Ordered n-tuples:

The extension of a 2-place predicate is a set of ordered *pairs*, $\langle \_, \_\rangle$.

The extension of a 3-place predicate is a set of ordered *triples*, $\langle \_, \_, \_\rangle$.

The extension of a 4-place predicate is a set of ordered *quadruples*, $\langle \_, \_, \_, \_\rangle$.

etc.

E.g.

UD: positive integers
Sxyz: x is the sum of y and z

- extension of S: $\{\langle 2, 1, 1\rangle$, $\langle 3, 1, 2\rangle$, $\langle 3, 2, 1\rangle$, $\langle 9, 5, 4\rangle$, ...$\}$

Note that the same member can be used more than once in an n-tuple. E.g., in $\langle 2, 1, 1\rangle$ above, 1 is used in both the second and third place.

Note also that *order is important.*

$\langle 3, 2, 1\rangle$ and $\langle 1, 2, 3\rangle$ are *not the same.*

$\langle 3, 2, 1\rangle$ is in the extension of S. However $\langle 1, 2, 3\rangle$ is not in the extension of S since 1 is not the sum of 2 and 3.

Example:

UD: positive integers
Sxyz: x is the sum of y and z
a: 4
b: 3
c: 1

Under this interpretation, the sentence 'Sabc' is *true*, since $\langle 4, 3, 1 \rangle$ is in the extension of S.

However 'Sbca' is false since $\langle 3, 1, 4 \rangle$ is not in the extension of S.

6.2 Interpretations of Quantified Sentences

6.2.1 Satisfaction (informally)

A member of the UD satisfies a formula if that member is in the set of things for which that formula is true.

Example

UD: positive integers
Ax: x is even
Bx: x is divisible by nine.

The set of members of the UD that satisfy the formula $(\sim Ax)$ is the set of members of the UD that are not in the extension of A, i.e., those members that are not even: {1, 3, 5, 7, 9, ...}.

The set of members of the UD that satisfy the formula $(Ax \,\&\, Bx)$ is the set of members of the UD that are both in the extension of A and in the extension of B, i.e., those members that are both even and divisible by nine: {18, 36, 54, 72, ...}.

The set of members of the UD that satisfy the formula $(Ax \vee Bx)$ is the set of members of the UD that are either in the extension of A or in the extension of B (either even or divisible by nine): {2, 4, 6, 8, 9, 10, 12, 14, 16, 18, ...}.

The set of members of the UD that satisfy the formula $(Ax \supset Bx)$ is the set of members of the UD that are either in the extension of ∼A or in the extension of B (not even or divisible by nine): {1, 3, 5, 7, 9, 11, 13, 15, 17, 18, 19, 21, 23, 25, 27, 29, 31, ...}.

The set of members of the UD that satisfy the formula $(Ax \equiv Bx)$ is the set of members of the UD that are either in both the extensions of A and B, or in neither the extension of A nor in the extension of B: {1, 3, 5, 7, 11, 13, 15, 17, 18, 19, 21, 23, 25, 29, 31, 33, 35, 36, 37, ...}.

6.2.2 Satisfaction and Quantification

Universally quantified sentences say that every member of the UD satisfies the formula they quantify over.

In other words, if we consider the sentence $(\forall x)(Ax \supset Bx)$, this sentence is saying that the set of members of the UD that satisfy the formula $(Ax \supset Bx)$ *is the same* as the set of *all members of the UD*.

In this case it is saying that the set {1, 3, 5, 7, 11, 13, 15, 17, 18, 19, 21, 23, 25, 29, 31, 33, 35, 36, 37, ...} *is the same as* the set of all members of the UD: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, ...}.

Clearly, the sentence is *false*.

Existentially quantified sentences say that at least one member of the UD satisfies the formula they quantify over.

If we consider the sentence $(\exists x)(Ax \ \& \ Bx)$, this sentence is saying that the set of members of the UD that satisfy the formula $(Ax \ \& \ Bx)$ is not empty, i.e., that it has at least one member.

In this case, it is saying that the set {18, 36, 54, 72, ...} is not empty. This is obviously true, and so is the sentence.

In general,

A Universally quantified sentence

- Is **true** on an interpretation when the set of things in the UD which satisfy the formula quantified over is just the set of everything in the UD.
- Is **false** on an interpretation when at least one member of the UD is not in the set of things which satisfy the formula quantified over.

An Existentially quantified sentence

- Is **true** on an interpretation if the set of things in the UD which satisfy the formula quantified over is not empty.
- Is **false** on an interpretation when the set of things in the UD which satisfy the formula quantified over is empty.

Example:

UD: positive integers
Ex: x is even
Sx: x is divisible by 7

$(\exists x)(Ex \ \& \ Sx)$ is true.

The set of things in the UD which satisfy $(Ex \ \& \ Sx)$ is the set of things which are both even and divisible by 7.

This set is not empty, since it includes, for instance, 14. So the sentence is true.

6.3 Quantificational Truth, Falsity, Indeterminacy

Consider the formula: $(\forall x)(Gx \supset \sim Ex)$.

Can we find an interpretation on which this sentence is true?

Here is one:

UD: automobiles
Gx: x runs on gasoline
Ex: x is an electric car

Are there any interpretations on which it is false?

Here is one:

UD: positive integers
Gx: x is greater than 0
Ex: x is even

Consider the formula: $(\forall x)(\forall y)(Gxy \supset (Gxy \vee Gyx))$.

This formula is true, for instance, on the following interpretation:

UD: positive integers
Gxy: x is greater than y

Can we find an interpretation that makes this sentence false?

⇒ **No.** No matter how hard we try we will never find an interpretation on which this sentence is false.

If you examine the sentence, this becomes obvious, for it says that any two members of the UD are such that, if they satisfy Gxy, then they satisfy (Gxy or Gyx) as well. This is clearly true no matter how we interpret the predicate G.

This sentence is therefore **quantificationally true**.

6.3.1 Definitions

A sentence, **P**, is:

Quantificationally True (q-true) iff[5] **P** is true no matter what the interpretation.

Quantificationally False (q-false) iff **P** is false no matter what the interpretation.

---

[5] 'iff' is a short way of writing 'if and only if'.

Quantificationally Indeterminate (q-indeterminate) iff **P** is neither q-true nor q-false.

6.3.2 Determining whether a sentence is q-true, q-false, or q-indeterminate

To show that a sentence **is q-true**, we must show that the sentence is true no matter what the interpretation.

Since it would in general be impossible to construct every conceivable interpretation for a sentence, in order to show that a sentence is q-true we must use *reasoning.*

Likewise, we must use reasoning to show that a sentence **is q-false**, since we must show that it is false no matter what the interpretation.

On the other hand, to show that a sentence **is not q-true**, it suffices to construct an interpretation on which the sentence is false.

This serves as a counter-example to the claim that the sentence is true on every interpretation.

If we can construct an interpretation on which it is false, this shows that the sentence is not true on every interpretation.

Similarly, to show that a sentence **is not q-false**, we construct an interpretation on which the sentence is true.

To show that a sentence **is q-indeterminate**, we must do 2 things:

1. We must construct an interpretation on which the sentence is true (this shows that the sentence is not q-false).
2. We must construct an interpretation on which the sentence is false (this shows that the sentence is not q-true).

In this way we show that the sentence is neither q-false nor q-true. This, however, is just what it means for a sentence to be q-indeterminate.

6.3.3 Reasoning Strategies

Here are some general strategies for showing q-truth and q-falsity for different sentence forms. We consider truth-functional compounds, universally quantified sentences, and existentially quantified sentences:

Sentence	**q-true**	**q-false**
$\mathbf{P}$	Show that **P** is q-true	Show that **P** is q-false.
$\sim \mathbf{P}$	Show that **P** is q-false.	Show that **P** is q-true.
$\mathbf{P} \,\&\, \mathbf{Q}$	Show that both conjuncts are q-true **OR** Show that $(\sim \mathbf{P} \vee \sim \mathbf{Q})$ is q-false.	Show that either **P** or **Q** is q-false **OR** Assume **P** is true on an interpretation, and show that **Q** cannot then be true (or vice-versa).
$\mathbf{P} \vee \mathbf{Q}$	Show that one of the disjuncts is q-true **OR** show that $(\sim \mathbf{P} \,\&\, \sim \mathbf{Q})$ is q-false.	Show that both **P** and **Q** are q-false **OR** Show that $(\sim \mathbf{P} \,\&\, \sim \mathbf{Q})$ is q-true.
$\mathbf{P} \supset \mathbf{Q}$	Assume **P** is true on an interpretation and show that **Q** must then be true **OR** Show that $(\mathbf{P} \,\&\, \sim \mathbf{Q})$ is q-false **OR** Show that **P** is q-false.	Assume **P** is true on an interpretation and show that **Q** cannot then be true.
$\mathbf{P} \equiv \mathbf{Q}$	1) Assume **P** is true on an interpretation and show that **Q** must then be true. 2) Assume **Q** is true on an interpretation and show that **P** must then be true	Assume **P** is true on an interpretation and show that **Q** cannot then be true **OR** Assume **Q** is true on an interpretation and show that **P** cannot then be true.

Sentence[6]	**q-true**	**q-false**
$(\forall x)\mathbf{P}$	Show that every member of the UD satisfies **P** no matter what the interpretation	Show that on any interpretation there is a member of the UD that does not satisfy **P**
$(\forall x)(\mathbf{P}\ \&\ \mathbf{Q})$	Show that both conjuncts are satisfied by every member of the UD	Show that on any interpretation there is a member of the UD that either does not satisfy **P** or does not satisfy **Q**
$(\forall x)(\mathbf{P} \vee \mathbf{Q})$	Show that every member of the UD is such that it will satisfy one or both of the disjuncts	Show that on any interpretation there is a member of the UD that satisfies neither **P** nor **Q**.
$(\forall x)(\mathbf{P} \supset \mathbf{Q})$	Assume **P** is satisfied by some member of the UD and show that **Q** must then be satisfied **OR** Show that $(\mathbf{P}\ \&\ \sim\mathbf{Q})$ can never be satisfied **OR** Show that **P** can never be satisfied.	Show that on any interpretation there is a member of the UD that satisfies **P** but not **Q**
$(\forall x)(\mathbf{P} \equiv \mathbf{Q})$	1) Assume **P** is satisfied and show that **Q** must then be satisfied. 2) Assume **Q** is satisfied and show that **P** must then be satisfied	Show that on any interpretation there is a member of the UD that satisfies **P** but not **Q**, or vice-versa.

---

[6] Note: In the two tables below we assume that both **P** and **Q** contain the variable **x**.

Sentence	**q-true**	**q-false**
$(\exists x)\mathbf{P}$	Show that on any interpretation there is a member that satisfies **P**	Show that no member of the UD can satisfy **P** no matter what the interpretation.
$(\exists x)(\mathbf{P}\ \&\ \mathbf{Q})$	Show that on any interpretation there is a member that satisfies both conjuncts	Show that one or both of **P** or **Q** can never be satisfied no matter the interpretation **OR** Assume **P** is satisfied and show that **Q** cannot then be satisfied (or vice-versa).
$(\exists x)(\mathbf{P} \vee \mathbf{Q})$	Show that on any interpretation there is a member that satisfies one or both of the disjuncts	Show that no member of the UD will satisfy either of the disjuncts.
$(\exists x)(\mathbf{P} \supset \mathbf{Q})$	Show that on any interpretation, there is a member such that it either does not satisfy **P** or it satisfies **Q**	Assume **P** is satisfied and show that **Q** cannot then be satisfied.
$(\exists x)(\mathbf{P} \equiv \mathbf{Q})$	Show that on any interpretation there is a member such that that member either satisfies both **P** and **Q** or else that member satisfies neither **P** nor **Q**	Assume **P** is satisfied and show that **Q** cannot then be satisfied or vice-versa

6.3.4 Examples

Example 1

Show that $(\forall x)(Bx \supset Cx)$ & $(\exists x)(Bx$ & $\sim Cx)$ is q-false.

Proof For the sentence to be true, both conjuncts must be true. So let us imagine that there is an interpretation on which the first conjunct is true. In that case, every member of the UD on the interpretation is such that, if it is in the extension of B, then it is also in the extension of C. So it can never be the case that a member of the UD is in the extension of B but not in the extension of C. Now look at the second conjunct. It says that there is such a member. So if we make the first conjunct true, then we cannot make the second conjunct true. Thus both conjuncts can never be true at the same time. The sentence is, therefore, q-false. □[7]

Example 2

Show that $(\forall x)(Px \supset Qx) \supset (\forall y)(\sim Py \vee Qy)$ is q-true.

Imagine that there is an interpretation on which $(\forall x)(Px \supset Qx)$ is true. Now one thing we know about every member of the UD is that it will either be in the extension of P or it will not (note also that either way it will not bar us from satisfying $(Px \supset Qx)$). Suppose that it is in the extension of P. In that case, since it satisfies $(Px \supset Qx)$ we know that it will also have to be in the extension of Q and thus satisfy Qy. Suppose, on the other hand, that it is not in the extension of P. In this case it will satisfy the formula $\sim Py$. So our member will either satisfy $\sim Py$ or Qy depending on whether it is in the extension of P, i.e., it will satisfy the formula $(\sim Py \vee Qy)$. Therefore the sentence $(\forall y)(\sim Py \vee Qy)$ must be true on the interpretation. So $(\forall x)(Px \supset Qx) \supset (\forall y)(\sim Py \vee Qy)$ is q-true. □

Example 3

Show that $(\forall x)(\exists y)(Pxy$ & $Pyx)$ is not q-true.

---

[7] The symbol □ signifies the end of a proof.

Here we must construct an interpretation on which the sentence is false.

Note that this is a universally quantified sentence, so we must construct an interpretation on which at least one member (call it x) does not satisfy the formula $(\exists y)(Pxy \ \& \ Pyx)$.

Let us pretend that our x is 1 (just to help our thought process along), with a UD of positive integers.

The rest of the sentence says that there is a y such that 1 bears some relation P to it, and such that it bears some relation P to 1. Since the sentence is existentially quantified we need to show that there is no such y.

What relation could satisfy these criteria? Here is one:

Pxy: x is greater than y.

So the sentence says that there is a y that is greater than 1, and that 1 is greater than it. This is impossible. So the sentence is false on this interpretation.

Since we have shown that there is at least one interpretation on which the sentence is not true, we can say that the sentence is not q-true.

Example 4

Show that: $(\exists x)Fxx \supset (\forall y) \sim Fyy$ is not q-false.

To show that this sentence is not q-false, we need to construct an interpretation on which it is true.

Note that the main connective is a material conditional. So it is true when either the antecedent is false or when the consequent is true.

Let us try and make the antecedent false!

If we choose:

UD: positive integers
Fxy: x is greater than y

Then the antecedent says that there is something in the UD that is greater than itself. Of course there is no such member. So no member of the UD can satisfy Fxx and we have found an interpretation on which the sentence $(\exists x)Fxx \supset (\forall y) \sim Fyy$ is true.

Example 5

Show that $(Aa$ & $Ba)$ & $(\forall x)(Cxa \supset\sim (Ax$ & $Bx))$ is q-indeterminate.

We need to construct one interpretation on which it is true and one on which it is false.

Let us start by finding one on which it is true. For this, we will need to make both conjuncts true.

We will use a UD of positive integers, and we will pick something to be a. Let us try using 2. If it does not work we will choose something else.

Now look at the first conjunct. It says that there are two things that are true of 2. So let us pick two things that we know are true of two:

Ax: x is even
Bx: x is prime

Now look at the second conjuct. It says that every number that bears the relation C to 2 is not both even and prime. Can we find such a relation? This one will work:

Cxy: x is greater than y

2 is the only even prime number, so any number greater than 2 will not be both even and prime.

Now let us construct an interpretation on which the sentence is false. This is much easier:

UD: living things
Ax: x is an animal
Bx: x is a plant
Cxy: x loves y
a: Benji (the dog)

6.4 Other quantificational properties

6.4.1 Quantificational Equivalence

P and **Q** are q-equivalent iff there is *no* interpretation on which **P** and **Q** have *different truth values.*

To show that two sentences are q-equivalent, we must **reason** about it.

Showing q-equivalence is very similar to showing that the sentence $(\mathbf{P} \equiv \mathbf{Q})$ is q-true.

First, assume that **P** is true on an interpretation and show that **Q** must then be true as well. Then assumne that **Q** is true on an interpretation and show that **P** must then be true as well.[8]

Example

Show that $(\exists x)(Ax \,\&\, Bx)$ and $\sim (\forall x)(Ax \supset\sim Bx)$ are q-equivalent.

Proof 1) Assume that $(\exists x)(Ax \,\&\, Bx)$ is true on an interpretation. Then there is a member of the UD that is in the extension of both A and B. Now consider the sentence $(\forall x)(Ax \supset\sim Bx)$. This sentence says that for every member of the UD, if it is in the extension of A, then it is not in the extension of B. But we know that, given our initial assumption, there is such a member. So $(\forall x)(Ax \supset\sim Bx)$ is false, or in other words, $\sim (\forall x)(Ax \supset\sim Bx)$ is true on this interpretation. 2) Assume that $\sim (\forall x)(Ax \supset\sim Bx)$ is true on an interpretation. This says that it is not the case that every member of the UD is such that if it is in the extension of A, then it is not in the extension of B. So there must be a member for which this is not true. In other words there must be a member that is both in the extension of A and in the extension of B. This is just what the first sentence says. So $(\exists x)(Ax \,\&\, Bx)$ is true on this interpretation. □

---

[8] You can also use the following procedure, which amounts to the same thing: First, assume that **P** is true on an interpretation and show that **Q** must then be true as well. Then assumne that **P** is false on an interpretation and show that **Q** must then be false as well.

To show that two sentences are not q-equivalent, we must construct an interpretation on which:

- one of the sentences is false
- the other sentence is true

Example

Show that $(\forall x)(Bxa \supset Ax)$ and $(\forall x)(Ax \supset Bxa)$ are not q-equivalent.

Let us try and make the first sentence true.

Let's choose our UD to be positive integers, as usual, and we will interpret our constant, a, to be 2 (usually a good choice).

UD: Positive integers
a: 2

So we are looking for a relation, B, such that any x that bears B to a will be an A. Here is one:

Bxy: x is divisible by 2
Ax: x is even

Any number divisible by 2 is even, so the first sentence is true.

But wait, there is a problem. The second sentence is true as well!

Let us try and see if we will get a better result if we use a different constant. Try a: 4.

Now it works. Any number divisible by 4 is even, but not every even number is divisible by 4.

So the two sentences are not equivalent, since on this interpretation the first one is true, while the second one is false.

6.4.2 Quantificational consistency

A set of sentences, Γ (pronounced 'gamma'), is q-consistent iff there is an interpretation on which every sentence in Γ is true. It is q-inconsistent if it is not q-consistent.

To show that a set of sentences is q-consistent, construct an interpretation on which they are all true.

Example

Show that the set $\{(\exists x)(Px \ \& \ Qx), (\exists x)(Px \ \& \ \sim Qx)\}$ is q-consistent.

Let us try:

UD: positive integers
Px: x is prime

Now we need to choose Q so that it can either be true or not of primes.

This way, both sentences can be true at the same time.

Choose: Qx: x is odd.

Since 2 is a prime number, it satisfies $(Px \ \& \ \sim Qx)$, and every other prime number satisfies $(Px \ \& \ Qx)$. So both sentences are true on this interpretation.

Example

Show that the set $\{(\forall x)(\sim Axb \supset (\exists y)Ayx), \sim (\exists x)Axa\}$ is q-consistent.

Let us start off:

UD: positive integers
a: 1
b: 2

How can we choose the predicate A so that both sentences turn out true?

Let us look at the second sentence. It seems easier.

It says that there is no member of the UD that bears a certain relation to 1.

This will work:

Axy: x is less than y.

So now the second sentence says that there is no positive integer less than 1. This is true. The first sentence says that if a positive number is not less than 2, then there is some positive number less than it. This is also true.

Example

Show that $\{(\exists x)(Ax \ \& \ Bx), (\forall x)(Bx \supset Cx), (\forall x)(Cx \supset \sim Ax)\}$ is q-inconsistent.

Here we must use our reason.

We must show that there is no interpretation on which all the sentences are true.

The procedure is very similar to showing that (**P** & **Q**) is q-false, except in general the number of conjuncts will not always be 2 (as in our example).

We will need to show either that one or more of the sentences is q-false, or if we cannot show this, then we must do the following:

Assume that one of the sentences is true on an interpretation, and show that as a result, one of the other sentences must turn out false on that interpretation.

Proof It is clear that none of the sentences is q-false. So let us assume $(\exists x)(Ax \;\&\; Bx)$ is true on an interpretation. Then there is a member of the UD such that it is in both the extension of A and in the extension of B. Now the second sentence says that any member of the UD that is in the extension of B must also be in the extension of C. So our member must be in the extension of C. The third sentence, now, says that any member that is in the extension of C is not in the extension of A. So our member must not be in the extension of A. But this contradicts our original assumption, so the third sentence must be false on the assumption that the first sentence is true. So all the sentences cannot be true at the same time. The set is therefore q-inconsistent. □

6.4.3 Quantificational entailment

A set of sentences, Γ, q-entails ($\models$) a sentence **P** iff there is no interpretation on which every member of Γ is true but **P** is false.

To show that a set q-entails a sentence, you must use reasoning.

It is similar to showing that a sentence of the form **P** $\supset$ **Q** is true, where in this case, **P** is the conjunction of the members of Γ.

E.g., if $\Gamma = \{S_1, S_2, S_3, S_4\}$ then to show that Γ entails the sentence C, we must show that $[(S_1 \ \& \ S_2) \ \& \ (S_3 \ \& \ S_4)] \supset C$.

We do this by assuming that $[(S_1 \ \& \ S_2) \ \& \ (S_3 \ \& \ S_4)]$ is true, and then showing that C must also be true.

Example

Show that $\Gamma = \{Aa, (\forall x)(Ax \supset (Bx \vee Cx)), (\forall x)(Bx \supset Dx), (\forall x)(Cx \supset Dx)\} \models Da$.

Proof We begin by assuming that every member of Γ is true. Now Aa tells us that a is in the extension of A. But $(\forall x)(Ax \supset (Bx \vee Cx))$ tells us that if anything is in the extension of A, then it is also in either the extension of B or in the extension of C. So a is in the extension of either B or C. Let us consider both options. Suppose a is in the extension of B. Then since $(\forall x)(Bx \supset Dx)$ is true, a must also be in the extension of D. On the other hand, suppose a is in the extension of C. Then since $(\forall x)(Cx \supset Dx)$ is true, a must again be in the extension of D. So either way, a must be in the extension of D. So Da must be true if every member of Γ is true. $\square$

To show that Γ does *not* entail ($\not\models$) a sentence, **P**, we must construct an interpretation on which:

- Every sentence in Γ is true
- **P** is false.

Example

Show that:

$\Gamma = \{(\exists x)(\exists y)(Axy \ \& \ Ayx), (\forall x)(\forall y)((Axy \ \& \ Ayx) \supset Bxy)\} \not\models (\forall x)(\forall y)Bxy$

We will use a UD of positive integers.

Now let us see if we can make the first sentence true.

We can if we use:

Axy: x is divisible by y.

In that case, it will be true that there is an x and a y such that $(Axy \ \& \ Ayx)$. Just choose x and y to be the same member of the UD.

Now let us try and make the second sentence true.

What does it say? That every x and y in the UD are such that, if x is divisible by y and y is divisible by x, then Bxy have some relation to one another.

What can we use for B?

We just mentioned it earlier:

Bxy: x and y are identical.

So now we have made every sentence in Γ true. What can we say about $(\forall x)(\forall y)Bxy$?

This says that everything in the UD is the same as everything else. This is clearly false.

So we have constructed an interpretation on which every sentence in Γ is true but $(\forall x)(\forall y)Bxy$ is false.

So Γ does not entail $(\forall x)(\forall y)Bxy$.

6.4.4 Quantificational validity

An argument is q-valid iff there is no interpretation on which every premise is true and the conclusion false.

q-validity is very similar to q-entailment. In fact, q-validity is just a special case of q-entailment, where Γ is the set of premises in an argument, and **P** is the conclusion of the argument.

To show that an argument is q-valid, we must use reasoning.

Example

Show that the following argument is q-valid:

1. $(\exists x)(Ax \vee Bx)$
2. $(\forall x)(Ax \supset Dx)$
3. $(\forall x)(Bx \supset Cx)$

$\therefore$ $(\exists x)(Cx \vee Dx)$

Proof Imagine that premise 1 is true on an interpretation. Then in that case there is a member of the UD that is either in the extension of A or in the extension of B. Suppose this member is in the extension of A. Then by premise 2, we know that it must also be in the extension of D. On the other hand, suppose it is in the extension of B. Then by premise 3, we know it will also be in the extension of C. So we know that our member will either be in the extension of C or in the extension of D. So it will be true that there is a member in either the extension of C or the extension of D. But this is what the conclusion says. □

To show that an argument is not q-valid:

- construct an interpretation on which all the premises are true and the conclusion is false.

Example

Show that the following argument is not q-invalid:

1. $(\forall x) \sim (Ax \& Bx)$
2. $\sim Aa$

$\therefore$ Ba

Let us start by choosing a constant. 1 is often a good choice, so let's try it:

UD: positive integers

a: 1

Let us make the premises true:

Ax: x is even
Bx: x is odd.

Now the first and second premises are true. But what about the conclusion? It is also true. This is not what we want. We would like to have all the premises true and the conclusion *false*.

Let us try other predicates. Prime numbers are often a good choice so let us start with that:

Ax: x is prime

That will make the second premise true. Now let's try to make the first premise true. Let us choose

Bx: x is divisible by ...

1? no. (every prime number is divisible by 1)

2? no. (2 is divisbible by 2)

3? no. (3 is divisible by 3)

4? Yes! No prime number is divisible by 4.

Now both premises are true but 1 is not divisible by 4 so the conclusion is false.

7 Predicate Logic: Derivations

The derivation system for PL is called PD (short for "predicate derivations").

PD inludes all the rules of SD, and adds four new rules of inference.

We will be using PD+, which includes all the rules of SD, SD+, and PD, plus one extra rule of replacement.

So there are 5 new rules for us to learn in all.

Before taking a look at the new rules, let us first see how we can use the rules of SD (or SD+) to construct derivations in PD+.

For example, let us try to derive $(\forall x)(Bx \ \& \ Cx) \supset (\exists x)Cx$ from the assumption that $(\forall x)(Bx \ \& \ Cx) \supset [(\exists x)Bx \ \& \ (\exists x)Cx]$

Our ultimate goal is in the form of a material conditional, so the best strategy to use is to start with a subderivation which assumes the antecedent of the material conditional that we want to derive.

We will then try to derive the consequent, and then use the conditional introduction rule to get our desired goal.

1	$(\forall x)(Bx \mathbin{\&} Cx) \supset [(\exists x)Bx \mathbin{\&} (\exists x)Cx]$	Assum
2	$(\forall x)(Bx \mathbin{\&} Cx)$	Assum / $\supset$ I (sub-goal: $(\exists x)Cx$)
3		

The main connective of the sentence on line 1 is $\supset$. It is of the form $\mathbf{P} \supset \mathbf{Q}$ where $\mathbf{P}$ is $(\forall x)(Bx \mathbin{\&} Cx)$ and $\mathbf{Q}$ is $[(\exists x)Bx \mathbin{\&} (\exists x)Cx]$.

The sentence on line 2 is just line 1's $\mathbf{P}$, so we can use conditional elimination on lines 1 and 2 to get line 1's $\mathbf{Q}$:

1	$(\forall x)(Bx \mathbin{\&} Cx) \supset [(\exists x)Bx \mathbin{\&} (\exists x)Cx]$	Assum
2	$(\forall x)(Bx \mathbin{\&} Cx)$	Assum / $\supset$I (sub-goal: $(\exists x)Cx$)
3	$(\exists x)Bx \mathbin{\&} (\exists x)Cx$	1,2 $\supset$E

The main logical operator of the sentence on line 3 is &, so we can use conjunction elimination to get $(\exists x)Cx$, which is our sub-goal.

This then allows us to reach our ultimate goal through conditional introduction in the next step:

1	$(\forall x)(Bx \mathbin{\&} Cx) \supset [(\exists x)Bx \mathbin{\&} (\exists x)Cx]$	Assum
2	$(\forall x)(Bx \mathbin{\&} Cx)$	Assum / $\supset$I (sub-goal: $(\exists x)Cx$)
3	$(\exists x)Bx \mathbin{\&} (\exists x)Cx$	1,2 $\supset$E
4	$(\exists x)Cx$	3 & E
5	$(\forall x)(Bx \mathbin{\&} Cx) \supset (\exists x)Cx$	2-4 $\supset$I

In order to know whether it is possible to use one of the rules of SD/SD+,[9] it is important to identify the form of the sentence you are working with.

For example, the sentence $(\forall x)(Ax \vee Bx) \vee (\forall x)(Cx \mathbin{\&} Dx)$ is a sentence of the form $\mathbf{P} \vee \mathbf{Q}$. It's main connective is $\vee$ and we can try to use disjunction elimination to derive some sentence $\mathbf{R}$ from it.

However, the sentence: $(\forall x)[(Ax \vee Bx) \vee (Cx \mathbin{\&} Dx)]$ is **not** a sentence of the form $\mathbf{P} \vee \mathbf{Q}$. It is, in fact, a sentence of the form $(\forall \mathbf{x})\mathbf{P}$. We cannot use disjunction elimination on this sentence.

Likewise for the other rules of SD+.

7.1 New Rules in PD+

PD+ introduces five new rules.

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[9] From now on I will only write SD+. SD+ includes all of the rules of SD. Similarly for PD and PD+

There are four new rules of inference:[10]

- ∀E: Universal elimination.
- ∀I: Universal introduction.
- ∃E: Existential elimination.
- ∃I: Existential introduction.

There is one new rule of replacement:[11]

- QN: Quantifier negation.

7.1.1 Universal Elimination (∀E)

Suppose that $(\forall x)Ax$ is true. What can we then say about every member of the UD?

That it will satisfy the formula Ax.

So, Aa will be true, Ab will be true, Ac will be true. And so on for all of the members of the UD.

Similarly, if $(\forall x)(Ax \ \& \ Bx)$ is true, then $(Aa \ \& \ Ba)$ will be true, $(Ab \ \& \ Ab)$ will be true, $(Ac \ \& \ Ac)$ will be true, and so on.

We can capture the relationship between these sentences with our first rule of inference:

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[10] Rules of inference apply *only* to whole sentences.

[11] Rules of replacement may apply to components of sentences or to whole sentences.

Universal Elimination (∀E)

	$(\forall \mathbf{x})\mathbf{P}$
▻	$\mathbf{P}(\mathbf{a}/\mathbf{x})$

Formally speaking, the rule allows us to derive a substitution instance of $(\forall \mathbf{x})\mathbf{P}$ from $(\forall \mathbf{x})\mathbf{P}$

For example,

1	$(\forall x)(Px \supset Qx)$	
2	$Pa \supset Qa$	1 ∀E

In fact, since every member of the UD is assumed to satisfy the formula, we may use any constant we like, i.e., we could have done the following instead:

1	$(\forall x)(Px \supset Qx)$	
2	$Pc \supset Qc$	1 ∀E

And we may repeat the rule over and over again as many times as we like:

1	$(\forall x)(Px \supset Qx)$	
2	$Pa \supset Qa$	1 ∀E
3	$Pf \supset Qf$	1 ∀E
4	$Pv \supset Qv$	1 ∀E

Our only restriction is that we must conform to the rules for valid substitution instances.

Recall the procedure for forming a valid substitution instance:

1. Drop the **initial** quantifier.
2. Replace all of the **freed** variables with **the** desired constant (the same one each time).

So, for instance, the following is a correct application of ∀E:

1	$(\forall x)(\forall y)(Axy \supset Bxc)$	
2	$(\forall y)(Aay \supset Bac)$	1 ∀E

But the following are not:

1	$(\forall x)(\forall y)(Axy \supset Bxc)$		
2	$(\forall x)(Axa \supset Bxc)$	1 ∀E	**WRONG!**

1	$(\forall x)(\forall y)(Axy \supset Bxc)$		
2	$(\forall y)(Aay \supset Bxc)$	1 ∀E	**WRONG!**

When to use Universal Elimination

Universal elimination is a very useful rule of inference that may be used at any point during the derivation. If you have any accessible universally quantified sentences, consider using ∀E to derive your goal or sub-goal.

Which constant to use when applying Universal Elimination

To know which constant to use, we must always keep in mind what our immediate sub-goal is.

For example, imagine you are asked to derive Dd from the following:

1	$(\forall x)(Bx \supset Cx)$	Assum
2	Bs	Assum
3	$Cs \supset Dd$	Assum
4		

Our ultimate goal is Dd.

We see, from line 3, that if we could derive Cs, then we could use conditional elimination to derive Dd.

Our sub-goal is now Cs. Notice that on line 2, we have the sentence Bs. If we could somehow derive $(Bs \supset Cs)$ then we could use conditional elimination in order to derive Cs.

Our sub-goal is now $(Bs \supset Cs)$. We can get this by using Universal Elimination on line 1.

Here is the derivation in full:

1	$(\forall x)(Bx \supset Cx)$	Assum
2	Bs	Assum
3	$Cs \supset Dd$	Assum
4	$Bs \supset Cs$	1 ∀E
5	Cs	2,4 ⊃E
6	Dd	3,5 ⊃E

7.1.2 Existential Introduction (∃I)

Suppose we know that Janice has received an A on her logic test: Aj.

Now if it is true that Janice received an A, then it is of course true that at least one person in the class got an A: $(\exists x)Ax$.

We can capture the relationship between these sentences with our second new rule of inference:

Existential Introdiction (∃I):

	$\mathbf{P}(\mathbf{a}/\mathbf{x})$
$\rhd$	$(\exists \mathbf{x})\mathbf{P}$

Formally speaking, this rule says that if we have a substitution instance of an existentially quantified sentence, we may derive the existentially quantified sentence from it.

For example,

1	$Paz \mathbin{\&} Qzba$	
2	$(\exists y)(Pyz \mathbin{\&} Qzby)$	1 ∃I

When to use Existential Introduction

Existential introduction is useful when your goal is to derive an existentially quantified sentence.

You can try to derive a substition instance of the sentence first, and then use ∃I on it to get your goal sentence.

For example:

Derive: $(\exists z)(Lza \vee Lzb)$

1	$(\forall x)(Ax \supset Lxa)$	Assum
2	$(\forall x)(Bx \supset Lxb)$	Assum
3	$(\forall x)(Ax \vee Bx)$	Assum
4		

Our ultimate goal is $(\exists z)(Lza \vee Lzb)$. If we had a sentence that looked something like

$$L\_a \vee L\_b,$$

where the blank could be filled with any constant, then we could apply existential introducion to this sentence.

Since it doesn't matter which constant fills in the blank, let's just choose a. So our sub-goal will be $Laa \vee Lab$.

This sub-goal is a disjunction. One way to derive it is to derive Laa and then use disjunction introduction to get $Laa \vee Lab$. Or we could derive Lab and do the same thing. Either one will do.

Now notice that line 3 says that everything either satisfies Ax or Bx. But line 1 says that if anything satisfies Ax then it also satisfies Lxa. And line 2 says that if anything satisfies Bx then it also satisfies Lxb.

So $Aa \vee Ba$ is true, and depending on whether Aa or Ba is true, Laa or Lab will be true as well. And in either case $Laa \vee Lab$ will be true as well.

1	$(\forall x)(Ax \supset Lxa)$	Assum
2	$(\forall x)(Bx \supset Lxb)$	Assum
3	$(\forall x)(Ax \vee Bx)$	Assum
4	$Aa \vee Ba$	3 $\forall$E
5	Aa	Assum / $\vee$E
6	$Aa \supset Laa$	1 $\forall$E
7	Laa	5,6 $\supset$E
8	$Laa \vee Lab$	7 $\vee$I
9	Ba	Assum / $\vee$E
10	$Ba \supset Lab$	2 $\forall$E
11	Lab	9,10 $\supset$E
12	$Laa \vee Lab$	11 $\vee$I
13	$Laa \vee Lab$	4,5-8,9-12 $\vee$E
14	$(\exists z)(Lza \vee Lzb)$	13 $\exists$I

7.1.3 Universal Introduction ($\forall$I)

Recall the way we reasoned (in english) to show why, for instance, a sentence is q-true.

For instance, to show that a sentence of the form $(\forall x)(\mathbf{P} \supset \mathbf{Q})$ is q-true, we:

1. Assume that there is *a member* of the UD that satisfies **P**.
2. Show that that member must satisfy **Q** as well.

The reason this is a good proof is that we did not assume anything in particular about that member of the UD when we were reasoning about it.

It could have been any member.

That is the point, by choosing an *arbitrary* member in this way, we show that no matter what member we choose, the same result will follow.

Our next new rule of inference functions similarly.

For instance, if we have derived Pa without making any assumptions about the properties of a, then we can universalize: we can infer that $(\forall x)Px$.

Here is the formal definition of the rule:

Universal Introduction (∀I):

$$\begin{array}{r|l} & \mathbf{P}(\mathbf{a}/\mathbf{x}) \\ \rhd & (\forall \mathbf{x})\mathbf{P} \end{array}$$

provided that:

1. **a** does not occur in an open assumption.
2. **a** does not occur in $(\forall x)\mathbf{P}$.

Before explaining the reason for these restrictions, let us consider an example.

Example: Derive $(\forall x)(Ax \supset Cxa)$

1	$(\forall x)(Ax \supset Bx)$	Assum
2	$(\forall x)(Bx \supset Cxa)$	Assum
3		

We want to derive a universally quantified sentence.

We can do it if we can derive a *substitution instance* of it that meets our criteria.

Something like: $(A_{\_} \supset C_{\_}a)$ where the blank is filled in with some constant.

Let's pick a constant. We should choose one that does not occur in either an open assumbion or in the universally quantified sentence we want to derive. We cannot pick a, so let us choose b:

$$(Ab \supset Cba)$$

This will be our subgoal. Since it is a material conditional we will assume the antecedent and try to derive the consequent.

1	$(\forall x)(Ax \supset Bx)$	Assum
2	$(\forall x)(Bx \supset Cxa)$	Assum
3	$\quad Ab$	Assum / $\supset$I
4	$\quad Ab \supset Bb$	1 $\forall$E
5	$\quad Bb$	3,4 $\supset$E
6	$\quad Bb \supset Cba$	2 $\forall$E
7	$\quad Cba$	5,6 $\supset$E
8	$Ab \supset Cba$	3-7 $\supset$I
9	$(\forall x)(Ax \supset Cxa)$	8 $\forall$I

Note that in step 9 we have not violated the first restriction on $\forall$I, even though the constant b occurs in an assumption at line 3. In this case, applying $\forall$I is legal since this assumption *is no longer open* at line 8, for line 3 is not within the scope of line 8.

On the other hand, here is an incorrect use of $\forall$I:

Derive: $(\forall x)(Ax \supset Bx)$

1	$Aa \& Ba$	Assum	
2	$\quad Aa$	Assum / $\supset$I	
3	$\quad Ba$	1 &E	
4	$Aa \supset Ba$	2-3 $\supset$I	
5	$(\forall x)(Ax \supset Bx)$	4 $\forall$I	**WRONG!**

In this case a occurs in an **open** assumption. Assumption 1 is open at line 4.

This is an example of violating our **first restriction.**

We need the first restriction restriction because we need it to be the case that the instantiating constant we universalize over is just some *arbitrary* member of the UD. But if we already have some prior knowledge of the member we choose to reason about, then that choice is not an arbitrary choice.

Another way to think about it is if we know something (represented by the prior assumptions) about a particular constant, then we may use that knowledge to derive something that we may not have been able to derive otherwise. But this should **not** be allowed, since we are trying to derive something about *any old member* of the UD.

For instance, suppose we want to prove that "Every human being has a mother". We could reason validly as follows: Pick any member of the UD - call this member Jane Doe. Since all human beings have been born at some point, then Jane must have been born at some point. If Jane was born at some point, then someone gave birth to her. A person who gives birth to a baby is the mother of that baby, by definition.[12] So Jane must have a mother. Therefore everyone has a mother. □

On the other hand, we could reason *invalidly* to the conclusion "All people has been to Washington DC." as follows: Consider Abraham Lincoln. Abraham Lincoln was president of the USA. But presidents of the USA live in Washington DC. So Abraham Lincoln has been to Washington DC. Therefore all people have been to Washington DC.

This is obviously invalid. The reason is that we already knew something about Lincoln (that he was president) which allowed us to reason to the conclusion that he has been to Washington DC. This would not have worked for just any member of the UD that we happened to pick.

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[12] Let us ignore, for the time being, surrogate mothers and other modern technology.

Returning to our previous two derivation examples, in the first example, *a* was introduced without making any particular assumptions about what it was. We could have chosen *b* or *c* or *t* or *u* instead or anything we wanted, and we still could have derived our conclusion.

In the second example, we used *our previous knowledge* of *a*'s properties (represented by the assumption in line 1) to derive something about it.

Our derivation would not have worked had we chosen *b* or *c*.

So it is illegitimate to universalize in this case.

As for the **second restriction**—that the constant we universalize over should not occur in the resulting universally quantified sentence—we need this restriction in order to block inferences of the following sort:

1	$(\forall x)Lxx$	Assum	
2	Lhh	1 ∀E	
3	$(\forall x)Lxh$	1 ∀I	**WRONG!**

Imagine we translate this as:

1. Everyone loves themselves.
2. Henry loves himself.
3. Everyone loves Henry.

(2) certainly follows from (1), but as nice as (3) would be for Henry if it were true, it certainly does not follow from (1) or (2)!

7.1.4 Existential Elimination (∃E)

This rule is the least intuitive of the four. This is partly because "Existential Elimination" is not a very good name for it. Unfortunately, we will just have to live with this.

The idea, however, is this. Suppose you know that some member of the UD satisfies a formula **P**. In other words, suppose you know that $(\exists x)\mathbf{P}$ is true.

Then what you can do is assume that a member, represented by an arbitrary constant, satisfies the formula (in a sub-derivation).

Now you will try to derive *something useful* from that assumption.

If what you derive meets certain criteria, then you are allowed to affirm it in your main derivation.

Why? Because you did not assume anything you didn't already know.

You knew that there was at least one member satisfying **P**.

The only extra thing you presupposed in your subderivation was that that member's name was "*a*" (or *b* or *c* or whatever).

As long as whatever you conclude from your subderivation makes no mention of *a*, then you may affirm it in your main derivation.

The full rule:

Existential Elimination (∃E):

$$\begin{array}{l|l} & (\exists \mathbf{x})\mathbf{P} \\ & \begin{array}{|l} \mathbf{P}(\mathbf{a}/\mathbf{x}) \\ \hline \mathbf{Q} \end{array} \\ \rhd & \mathbf{Q} \end{array}$$

provided that:

1. **a** does not occur in an open assumption.
2. **a** does not occur in $(\exists x)\mathbf{P}$.
3. **a** does not occur in **Q**.

Let us look at an example of a correct application of ∃E:

Example:

1	$(\forall x)(Bbx \supset Cx)$	Assum
2	$(\exists x)Bbx$	Assum
3	$\quad Bba$	Assum / ∃E
4	$\quad Bba \supset Ca$	1 ∀E
5	$\quad Ca$	3,4 ⊃E
6	$\quad (\exists x)Cx$	5 ∃I
7	$(\exists x)Cx$	2,3-6 ∃E

We have obeyed all three restrictions: *a* in line 3 does not occur in an open assumption, or in the existentially quantified sentence that line 3 is a substitution instance of (line 2). Finally, *a* does not appear in line 7, where we actually used the ∃E rule.

Let us try and understand the reasons for these three restrictions. The **first restriction** is similar to the first restriction in ∀I.

In ∃E, we begin by knowing that *some* member of the UD satisfies the formula **P**, *but we don't know which one.*

So we should not make use of anything we know about any particular member of the UD when reasoning about the member that satisfies **P**.

E.g.,

1	$Ca \supset Ba$	Assum	
2	$(\exists x)Cx$	Assum	
3	$\quad Ca$	Assum / ∃E	
4	$\quad Ba$	1,3 ⊃E	
5	Ba	2,3-4 ∃E	**WRONG!**

We *do not know which* member of the UD satisfies Cx. So the name that we choose to represent it should be arbitrary. It should not be the name of a member that we have prior knowledge of. If it was, then our choice *would not be arbitrary*—it would not represent just any old member of the UD.

In this example we do indeed have prior knowledge of a. We know from assumption 1 that if a is a C, then it is a B as well.

The reason for the **second restriction** is similar.

If we know that *something* bears a relation to a particular member of the UD, we shouldn't assume that the member that bears this relation to that particular member is the particular member itself!

There is no reason to assume that the member that bears the relation to that particular member is the particular member itself. So the choice is not arbitrary.

As for the **third** restriction, unlike the first two restrctions which restrict the knowledge that you are allowed to bring into a sub-derivation, this restriction limits what knowledge you can *take out.*

In your sub-assumption, you assumed that an arbitrary member of the UD satisfied **P** and then reasoned about what other properties this arbitrary member would have *if* it satisfied **P**.

But it doesn't follow that the thing that you picked *actually* has the specified property.

E.g.,

1	$(\forall x)(Cx \supset Nx)$	Assum	
2	$(\exists x)(Px \,\&\, Cx)$	Assum	
3	$\vert\ Pa \,\&\, Ca$	Assum / ∃E	
4	$\vert\ Ca$	3 &E	
5	$\vert\ Ca \supset Na$	1 ∀E	
6	$\vert\ Na$	4,5 ⊃E	
7	Na	2,3-6 ∃E	**WRONG!**

Imagine that:

UD: everything
Px: x is a person
Cx: x plays for the Montreal Canadiens
Nx: x plays in the NHL

This argument goes as follows:

Of course, everyone who plays for the Montreal Canadiens plays in the NHL. Well we know that some people play for the Montreal Canadiens. Suppose, just for the sake of argument, that that someone is, oh, say, me. Well if I were to play for the Canadiens, then I would have to play in the NHL also.

Therefore, I play in the NHL. **No!**

However, this argument is ok:

1	$(\forall x)(Cx \supset Nx)$	Assum
2	$(\exists x)(Px \,\&\, Cx)$	Assum
3	$\quad Pa \,\&\, Ca$	Assum / $\exists$E
4	$\quad Ca$	3 &E
5	$\quad Ca \supset Na$	1 $\forall$E
6	$\quad Na$	4,5 $\supset$E
7	$\quad (\exists x)Nx$	6 $\exists$I
8	$(\exists x)Nx$	2,3-7 $\exists$E

We *can* conclude based on this argument that *someone* plays in the NHL.

Here is another (correct) example.

1	$(\exists x)Bx \supset (\forall y)Cy$	Assum
2	$(\forall x)(Ax \supset Bx)$	Assum
3	$(\exists x)Ax$	Assum
4	$\quad Aa$	Assum / $\exists$E
5	$\quad Aa \supset Ba$	2 $\forall$E
6	$\quad Ba$	4,5 $\supset$E
7	$\quad (\exists x)Bx$	6 $\exists$I
8	$\quad (\forall y)Cy$	1,7 $\supset$E
9	$(\forall y)Cy$	3,4-8 $\exists$E

The point of this example is to show you that the **Q** that you take into your main derivation from your sub-deivation need not look anything like your original existentially quantified sentence. Notice that the sentence on line 3 and the sentence on line 9 look nothing like one another.

(This is similar to the $\vee$E rule.)

When to use Existential Elimination

1. Whenever you have an accessible existentially quantified sentence, consider using it to achieve your goal or sub-goal.
    - Make sure the instantiating constant that you use does not occur within an open assumption, the existentially quantified sentence, or the goal or sub/goal you are trying to achieve.
2. If you can derive a contradiction within an existential elimination subderivation, you can use it to derive any sentence you like.

Here is an example in which the 2^{nd} strategy is used:

Derive $\sim (\exists x)(Px \ \& \ Qx)$ from the assumption $(\forall x)(Px \supset \sim Qx)$.

Note that $(\forall x)(Px \supset \sim Qx)$ is an example of an E-sentence, and $\sim (\exists x)(Px \ \& \ Qx)$ is an example of a negation of an I-sentence, so these two sentences should be equivalent.

1	$(\forall x)(Px \supset\sim Qx)$	Assum
2	$(\exists x)(Px \,\&\, Qx)$	Assum / ∼I
3	$Pa \,\&\, Qa$	Assum / ∃E
4	Pa	3 & E
5	$Pa \supset\sim Qa$	1 ∀E
6	$\sim Qa$	4,5 ⊃E
7	Qa	3 &E
8	$(\exists x)(Px \,\&\, Qx)$	Assum / ∼I
9	Qa	7R
10	$\sim Qa$	6R
11	$\sim (\exists x)(Px \,\&\, Qx)$	8-10 ∼I
12	$\sim (\exists x)(Px \,\&\, Qx)$	2,3-11 ∃E
13	$(\exists x)(Px \,\&\, Qx)$	2R
14	$\sim (\exists x)(Px \,\&\, Qx)$	2-13 ∼I

7.1.5 Quantifier Negation (QN)

According to the rule of quanitifier negation, any sentence of the form $\sim (\forall x)\mathbf{P}$ can be replaced with a sentence of the form $(\exists x) \sim \mathbf{P}$, and any sentence of the form $\sim (\exists x)\mathbf{P}$ can be replaced with a sentence of the form $(\forall x) \sim \mathbf{P}$.

Clearly, if not all members of the UD satisfy **P**, then there is at least one member of the UD that does not satisfy **P** (and vice versa):

$$\sim (\forall x)\mathbf{P} \Leftrightarrow (\exists x) \sim \mathbf{P}$$

Equally clearly, if no members of the UD satisfy **P**, then all members of the UD do not satisfy **P** (and vice versa):

$$\sim (\exists x)\mathbf{P} \Leftrightarrow (\forall x) \sim \mathbf{P}$$

Note that Quantifier Negation is a rule of *replacement*, not a rule of inference. Thus we can use it on parts of sentences as well as on whole sentences. E.g.

1	$Fa \supset \sim (\exists x)Gx$	Assum
2	$Fa \supset (\forall x) \sim Gx$	1 QN

Combined with Double Negation (DN), Quantifier Negation is a very powerful rule. E.g.,

1	$(\forall x)Px$	Assum
2	$\sim\sim (\forall x)Px$	1 DN
3	$\sim (\exists x) \sim Px$	2 QN

7.2 Establishing Syntactic Properties

7.2.1 Derivability

A sentence **P** of PL is derivable in PD+ ($\vdash$) from a set Γ of sentences of PL iff there is a derivation in PD+ in which all the primary assumptions are members of Γ and **P** occurs within the scope of only the primary assumptions.

To show that a sentence, **P**, is derivable from a set of sentences, $\Gamma = \{\mathbf{Q}, \mathbf{R}, \mathbf{S}, ...\}$:

- construct a derivation in which each member of Γ is a main assumption, and try to derive **P** within the outermost scope of the derivation.
- By "derive within the outermost scope" we mean that it is not good enough to derive **P** within the scope of a sub-assumption. **P** must occur as far "to the left" as possible.

Example

Show that $\{(\forall x)(Ax \supset \sim Bxa), (\exists x)Ax, (\forall x)(Bxa \vee Dx)\} \vdash (\exists x)(Dx \vee Cxb)$

We must try to derive $(\exists x)(Dx \vee Cxb)$ from:

1	$(\forall x)(Ax \supset \sim Bxa)$	Assum
2	$(\exists x)Ax$	Assum
3	$(\forall x)(Bxa \vee Dx)$	Assum
4		

Here is the derivation:

1	$(\forall x)(Ax \supset \sim Bxa)$	Assum
2	$(\exists x)Ax$	Assum
3	$(\forall x)(Bxa \vee Dx)$	Assum
4	Ac	Assum / ∃E
5	$Ac \supset \sim Bca$	1 ∀E
6	$\sim Bca$	4,5 ⊃E
7	$Bca \vee Dc$	3 ∀E
8	Dc	7,6 DS
9	$Dc \vee Ccb$	8 ∨I
10	$(\exists x)(Dx \vee Cxb)$	9 ∃I
11	$(\exists x)(Dx \vee Cxb)$	2,4-10 ∃E

7.2.2 Validity

An argument of PL is valid in PD+ iff the conclusion of the argument is derivable in PD+ from the set consisting of the premises. An argument of PL is *invalid* in PD+ iff it is not valid in PD+.

So if we have an argument of the form:

1. Premise 1
2. Premise 2

∴ Conclusion

To show that it is valid, we must show that:

$\{$Premise 1, Premise 2$\} \vdash$ Conclusion.

Example 1

1. $(\forall x)(Fax \supset Gbx)$
2. $(\forall x)(Hcx \supset Gbx)$

$\therefore$ $(\forall x)((Fax \vee Hcx) \supset Gbx)$

So we must derive $(\forall x)((Fax \vee Hcx) \supset Gbx)$ from:

1	$(\forall x)(Fax \supset Gbx)$	Assum
2	$(\forall x)(Hcx \supset Gbx)$	Assum
3		

Here is the derivation:

1	$(\forall x)(Fax \supset Gbx)$	Assum
2	$(\forall x)(Hcx \supset Gbx)$	Assum
3	$Fad \vee Hcd$	Assum / $\supset$I
4	Fad	Assum / $\vee$E
5	$Fad \supset Gbd$	1 $\forall$E
6	Gbd	4,5 $\supset$E
7	Hcd	Assum / $\vee$E
8	$Hcd \supset Gbd$	1 $\forall$E
9	Gbd	6,7 $\supset$E
10	Gbd	3,4-6,7-9 $\vee$E
11	$(Fad \vee Hcd) \supset Gbd$	3-8 $\supset$I
12	$(\forall x)((Fax \vee Hcx) \supset Gbx)$	9 $\forall$I

Example 2

$$\begin{array}{ll} 1. & (\forall x)(Px \supset (\exists y)Gxy) \\ \hline \therefore & \sim (\exists x)(Px \ \& \ \sim (\exists y)Gxy) \end{array}$$

So we must derive: $\sim (\exists x)(Px \ \& \ \sim (\exists y)Gxy)$ from:

$$\begin{array}{r|ll} 1 & (\forall x)(Px \supset (\exists y)Gxy) & \text{Assum} \\ \cline{2-2} 2 & & \end{array}$$

Here is the derivation:

1	$(\forall x)(Px \supset (\exists y)Gxy)$	Assum
2	$(\exists x)(Px \ \& \ \sim (\exists y)Gxy)$	Assum / ∼I
3	$Pa \ \& \ \sim (\exists y)Gay$	Assum / ∃E
4	$Pa \supset (\exists y)Gay$	1 ∀E
5	Pa	3 &E
6	$(\exists y)Gay$	4,5 ⊃E
7	$\sim (\exists y)Gay$	3 &E
8	$(\exists x)(Px \ \& \ \sim (\exists y)Gxy)$	Assum / ∼I
9	$(\exists y)Gay$	7R
10	$\sim (\exists y)Gay$	4R
11	$\sim (\exists x)(Px \ \& \ \sim (\exists y)Gxy)$	8-10 ∼I
12	$\sim (\exists x)(Px \ \& \ \sim (\exists y)Gxy)$	2, 3-11 ∃E
13	$(\exists x)(Px \ \& \ \sim (\exists y)Gxy)$	2R
14	$\sim (\exists x)(Px \ \& \ \sim (\exists y)Gxy)$	2-13 ∼I

7.2.3 Theorems

A sentence **P** of PL is a *theorem in* PD iff **P** is derivable in PD from the empty set.

In other words, we must derive **P** without making any assumptions; i.e., we must show that

$$\varnothing \vdash \mathbf{P}$$

(Note: $\varnothing$ just means the set with no members: $\{\}$).

Example 1

Show that: $(\forall x)((Px \,\&\, Qx) \supset (Rx \,\&\, Sax)) \supset (\forall y)(Py \supset (Qy \supset (Ry \,\&\, Say)))$ is a theorem in PD+.

To show this, we must derive $(\forall x)((Px \,\&\, Qx) \supset (Rx \,\&\, Sax)) \supset (\forall y)(Py \supset (Qy \supset (Ry \,\&\, Say)))$ from a derivation without any main assumptions:

1	$\mid$	

Here is the derivation:

1	$\mid\ \mid\ \underline{(\forall x)((Px \,\&\, Qx) \supset (Rx \,\&\, Sax))}$	Assum / $\supset I$
2	$\mid\ \mid\ \mid\ \underline{Pb}$	Assum / $\supset I$
3	$\mid\ \mid\ \mid\ \mid\ \underline{Qb}$	Assum / $\supset I$
4	$\mid\ \mid\ \mid\ \mid\ (Pb \,\&\, Qb) \supset (Rb \,\&\, Sab)$	$1\forall E$
5	$\mid\ \mid\ \mid\ \mid\ Pb \,\&\, Qb$	2,3$\&I$
6	$\mid\ \mid\ \mid\ \mid\ Rb \,\&\, Sab$	4,5 $\supset E$
7	$\mid\ \mid\ \mid\ Qb \supset (Rb \,\&\, Sab)$	3-6 $\supset I$
8	$\mid\ \mid\ Pb \supset (Qb \supset (Rb \,\&\, Sab))$	2-7 $\supset I$
9	$\mid\ \mid\ (\forall y)(Py \supset (Qy \supset (Ry \,\&\, Say)))$	$8\forall I$
10	$\mid\ (\forall x)((Px \,\&\, Qx) \supset (Rx \,\&\, Sax)) \supset (\forall y)(Py \supset (Qy \supset (Ry \,\&\, Say)))$	1-9 $\supset I$

Example 2

Show that

$$(\forall x)[(\exists y)(Pxy \supset (Ra \vee Qyx)) \supset (\exists z)(\sim Pxz \vee (Ra \vee Qzx))]$$

is a theorem in PD+.

Here is the derivation:

1	$(\exists y)(Pby \supset (Ra \vee Qyb))$	Assum / $\supset$I
2	$Pbc \supset (Ra \vee Qcb)$	Assum / $\exists$E
3	$\sim Pbc \vee (Ra \vee Qcb)$	2 Impl.
4	$(\exists z)(\sim Pbz \vee (Ra \vee Qzb))$	3$\exists$I
5	$(\exists z)(\sim Pbz \vee (Ra \vee Qzb))$	2-4 $\exists$E
6	$(\exists y)(Pby \supset (Ra \vee Qyb)) \supset (\exists z)(\sim Pbz \vee (Ra \vee Qzb))$	1-5 $\supset$I
7	$(\forall x)(\exists y)(Pxy \supset (Ra \vee Qyx)) \supset (\exists z)(\sim Pxz \vee (Ra \vee Qzx))$	6 $\forall$I

7.2.4 Equivalence

Sentences **P** and **Q** of PL are equivalent in PD+ iff **Q** is derivable in PD from {**P**} and **P** is derivable in PD from {**Q**}.

Example:

Show that $(\exists x)(\exists y)(Pxy \ \& \sim Qxy)$ and $\sim (\forall x)(\forall y)(Pxy \supset Qxy)$ are equivalent sentences of PL.

So we need to derive $\sim (\forall x)(\forall y)(Pxy \supset Qxy)$ from

1	$(\exists x)(\exists y)(Pxy \;\&\; \sim Qxy)$	Assum
2		

And then once we have done this, we must derive $(\exists x)(\exists y)(Pxy \;\&\; \sim Qxy)$ from

1	$\sim (\forall x)(\forall y)(Pxy \supset Qxy)$	Assum
2		

Here are the derivations:

1	$\vert\ (\exists x)(\exists y)(Pxy\ \&\ \sim Qxy)$	Assum
2	$\vert\ \vert\ (\forall x)(\forall y)(Pxy \supset Qxy)$	Assum / ~I
3	$\vert\ \vert\ \vert\ (\exists y)(Pay\ \&\ \sim Qay)$	Assum / ∃E
4	$\vert\ \vert\ \vert\ \vert\ Pab\ \&\ \sim Qab$	Assum / ∃E
5	$\vert\ \vert\ \vert\ \vert\ Pab$	4&E
6	$\vert\ \vert\ \vert\ \vert\ \sim Qab$	4&E
7	$\vert\ \vert\ \vert\ \vert\ (\forall y)(Pay \supset Qay)$	2∀E
8	$\vert\ \vert\ \vert\ \vert\ Pab \supset Qab$	7 ∀E
9	$\vert\ \vert\ \vert\ \vert\ Qab$	5,8 ⊃E
10	$\vert\ \vert\ \vert\ \vert\ \vert\ (\forall x)(\forall y)(Pxy \supset Qxy)$	Assum / ~I
11	$\vert\ \vert\ \vert\ \vert\ \vert\ Qab$	9R
12	$\vert\ \vert\ \vert\ \vert\ \vert\ \sim Qab$	6R
13	$\vert\ \vert\ \vert\ \vert\ \sim (\forall x)(\forall y)(Pxy \supset Qxy)$	10-12 ~I
14	$\vert\ \vert\ \vert\ \sim (\forall x)(\forall y)(Pxy \supset Qxy)$	3, 4-13 ∃E
15	$\vert\ \vert\ \sim (\forall x)(\forall y)(Pxy \supset Qxy)$	1, 3-14 ∃E
16	$\vert\ \vert\ (\forall x)(\forall y)(Pxy \supset Qxy)$	2R
17	$\vert\ \sim (\forall x)(\forall y)(Pxy \supset Qxy)$	2-16 ~I

For our second derivation, we will take advantage of the rules of SD+ and PD+ to save a few steps:

1	$\sim (\forall x)(\forall y)(Pxy \supset Qxy)$	Assum
2	$(\exists x) \sim (\forall y)(Pxy \supset Qxy)$	1 QN
3	$(\exists x)(\exists y) \sim (Pxy \supset Qxy)$	2 QN
4	$(\exists x)(\exists y) \sim (\sim Pxy \vee Qxy)$	3 Impl
5	$(\exists x)(\exists y)(\sim\sim Pxy \;\&\; \sim Qxy)$	4 DeM
6	$(\exists x)(\exists y)(Pxy \;\&\; \sim Qxy)$	5 DN

General rule of thumb for when to use the rules of SD+/PD+: Think of the rules of SD+/PD+ as "shortcuts". Focus on solving problems primarily with the rules of SD/PD and use the strategies that are explained in section 10.2 and 5.3 of the textbook. Then, at each step, if you think it is possible to use one of the "shortcut" rules of SD+/PD+, do so. Use trial and error. Remember, practice makes perfect.

7.2.5 Inconsistency

A set Γ of sentences of PL is inconsistent in PD iff there is a sentence **P** such that **P** and ∼ **P** are derivable in PD from Γ. A set Γ is consistent in PD iff it is not inconsistent in PD.

To show that a set of sentences is inconsistent, construct a derivation in which each member of the set is a main assumption, and derive a contradiction.

Rule of thumb: Use one of your main assumptions (preferably one whose main operator is a negation) as the assumption of a subderivation and use negation introduction to try to derive its opposite. You then will have your contradiction within the scope of the main derivation.

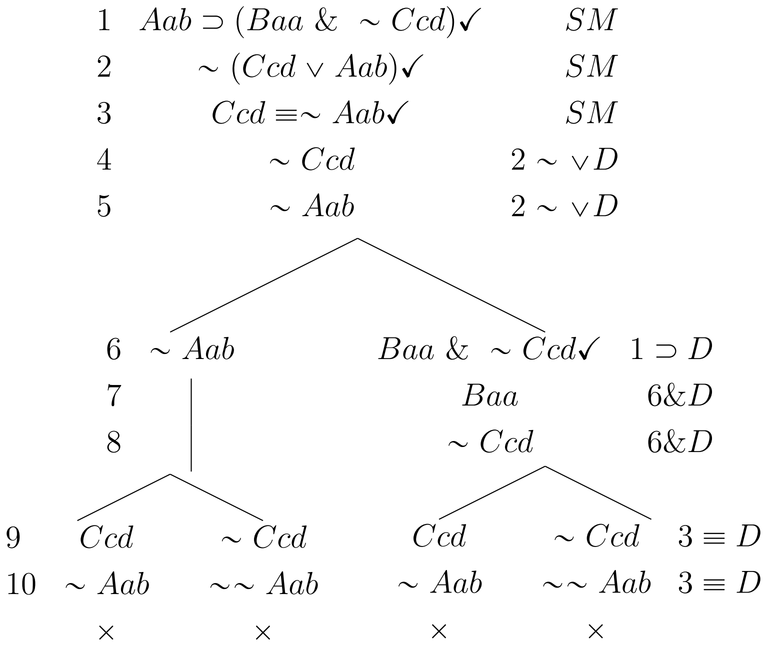
Example:

Show that the set $\{(\forall x)[(Ax \vee Bx) \supset Cx], (\forall x)(Cx \supset\sim (\exists y)Dy), (\exists x)(Ax \,\&\, Dx), (\exists x)(Bx \,\&\, Dx)\}$

1	$(\forall x)[(Ax \vee Bx) \supset Cx]$	Assum
2	$(\forall x)(Cx \supset\sim (\exists y)Dy)$	Assum
3	$(\exists x)(Ax \,\&\, Dx)$	Assum
4	$(\exists x)(Bx \,\&\, Dx)$	Assum
5	$(\exists x)(Bx \,\&\, Dx)$	Assum / ∼I
6	$Aa \,\&\, Da$	Assum / ∃E
7	Aa	6 &E
8	Da	6 &E
9	$Aa \vee Ba$	7 ∨I
10	$(Aa \vee Ba) \supset Ca$	1 ∀E
11	Ca	9,10 ⊃E
12	$Ca \supset\sim (\exists y)Dy$	2 ∀E
13	$\sim (\exists y)Dy$	11,12 ⊃E
14	$(\forall y) \sim Dy$	13 QN
15	$\sim Da$	14 ∀E
16	$(\exists x)(Bx \,\&\, Dx)$	Assum / ∼I
17	Da	8R
18	$\sim Da$	15R
19	$\sim (\exists x)(Bx \,\&\, Dx)$	16-18 ∼I
20	$\sim (\exists x)(Bx \,\&\, Dx)$	5, 6-19 ∃E
21	$(\exists x)(Bx \,\&\, Dx)$	5R
22	$\sim (\exists x)(Bx \,\&\, Dx)$	5-21 ∼I
23	$(\exists x)(Bx \,\&\, Dx)$	4R

8 Predicate Logic: Truth Trees

Here is an example of a truth tree for PL which can be decomposed using only the truth tree rules of SL:[13]



Unfortunately, the truth-tree rules for SL will not allow us to decompose universally or existentially quantified sentences of PL (or their negations).

For instance, the following two sentences are q-inconsistent, and a tree containing these two sentences as set members should close. But we cannot show this using only the rules of SL:

---

[13] Note: I do not have the proper software for drawing nice truth trees such as you find in the textbook. I have done my best, however the result is not always perfect. If the notation confuses you, please let me know.

1 $(\forall x)(Ax \supset Bx)$ SM

2 $(\exists x)(Ax \,\&\, Bx)$ SM

We need new rules. We will introduce four new rules in all.

8.1 Universal Decomposition ($\forall D$)

Universal decomposition allows you to replace a universally quantified sentence with a substitution instance of that sentence.

$$\begin{array}{c} (\forall \mathbf{x})\mathbf{P} \\ | \\ \mathbf{P}(\mathbf{a}/\mathbf{x}) \end{array}$$

The justification for this rule is similar to the justification for the Universal Elimination rule for derivations. A universally quantified sentence says that all members of the UD satisfy the formula quantified over. So if this is true then any particular member that you pick will satisfy the formula as well.

Note that when we decompose a universally quantified sentence **we do not check it off**. A substitution of the universally quantified sentence will be true no matter which instantiating constant is used, so we should be able to to instantiate the universally quantified sentence multiple times.

Note also that it is not always necessary to decompose a universally quantified sentence on every branch that passes through the universally quantified sentence. You may apply $\forall D$, if you wish, to particular branches only. The reason for this is partly so that you can save space and partly because it is always possible to apply it again to those branches that you missed the first time if you choose to do so later. (We will change this statement somewhat when we talk about *systematic trees* later).

8.2 Existential Decomposition ($\exists D$)

Existential decomposition allows you to replace an existentially quantified sentence with a substitution instance of that sentence.

$$\begin{array}{c}(\exists \mathbf{x})\mathbf{P}\checkmark \\ | \\ \mathbf{P}(\mathbf{a}/\mathbf{x})\end{array}$$

There is one restriction, however: the instantiating constant must be **foreign to the branch**. This restriction is similar to the first restriction of the Existential Elimination rule for derivations.

A branch in a truth tree represents one of a few possible scenarios that could be true, given our sentence members. Within a particular branch, the components of that branch represent what we know to be true for that particular scenario. An existentially quantified sentence on a branch tells us that *some* member of the UD satisfies the formula quantified over. But we do not know which member that is. We should not assume that that member is one for which we already have some knowledge.

Existential Decomposition is unlike Universal Decomposition and like the other truth tree rules in that once we decompose an existentially quantified sentence, we **check it off**. This is because an existentially quantified sentence tells us that at least one member of the UD satisfies the formula quantified over, but it does not tell us whether or not there are any other members which satisfy the formula. So we should not be allowed to apply it again.

8.3 Negated Universal Decomposition and Negated Existential Decomposition

Negated Universal Decomposition ($\sim \forall D$) allows you to replace a negated universally quantified sentence with an existentially quantified sentence. The formula quantified over in the existentially quantified sentence will be the negation of the formula quantified over in the original sentence.

Similarly, Negated Existential Decomposition ($\sim \exists D$) allows you to replace a negated existentially quantified sentence with a universally quantified sentence. The formula quantified over in the universally quantified sentence will be the negation of the formula quantified over in the original sentence.

These rules are similar to the quantifer negation (QN) for derivations.

$$\begin{array}{c} \underline{\sim \forall \mathbf{D} :} \\ \sim (\forall \mathbf{x})\mathbf{P}\checkmark \\ | \\ (\exists \mathbf{x}) \sim \mathbf{P} \end{array} \qquad\qquad \begin{array}{c} \underline{\sim \exists \mathbf{D} :} \\ \sim (\exists \mathbf{x})\mathbf{P}\checkmark \\ | \\ (\forall \mathbf{x}) \sim \mathbf{P} \end{array}$$

Here is an example using the new rules:

1	$(\forall x)(Bxa \vee Cx)$		SM
2	$\sim (\forall x)Cx$✓		SM
3	$(\exists x) \sim Cx$✓		$2 \sim \forall D$
4	$\sim Cb$		$3\exists D$
5	$Bba \vee Cb$✓		$1\forall D$
6	Bba	Cb	$5 \vee D$
		×	

Note that in line 4, in accordance with the restriction on Existential decomposition, we instantiated b and not a, since a already occurs on the branch at line 1.

Question: Are we done?

We have decomposed all of the sentences, however not all of our compound sentences have been checked off.

According to the criteria for a completed open branch in SL, it appears, then, that we are not done.

However it should be clear that nothing we can do will allow us to close all of the branches (try it). It should also be clear that nothing we can do will allow us to turn our open branches into completed open branches according to the criteria for SL (from chapter 4).

To help us answer this question, let us define some terms.

8.4 Definitions

I. A **literal** is an atomic sentence or the negation of an atomic sentence. E.g.: $Aa, \sim Bac, Sabc$, etc.

II. A **closed branch** is a branch that contains both a *literal* and its negation.

The following branch is *not* closed:

$$\vdots$$

21	$(\forall x)Px$
22	$\sim (\forall x)Px$

Remember we must have both a *literal* and its negation on the branch in order to close it. This is not the case here. However, we can close it easily in a few more steps:

$$\vdots$$

21	$(\forall x)Px$	
22	$\sim (\forall x)Px✓$	
23	$(\exists x) \sim Px✓$	$22 \sim \forall D$
24	$\sim Pa$	$23\exists D$
25	Pa	$21\forall D$
	×	

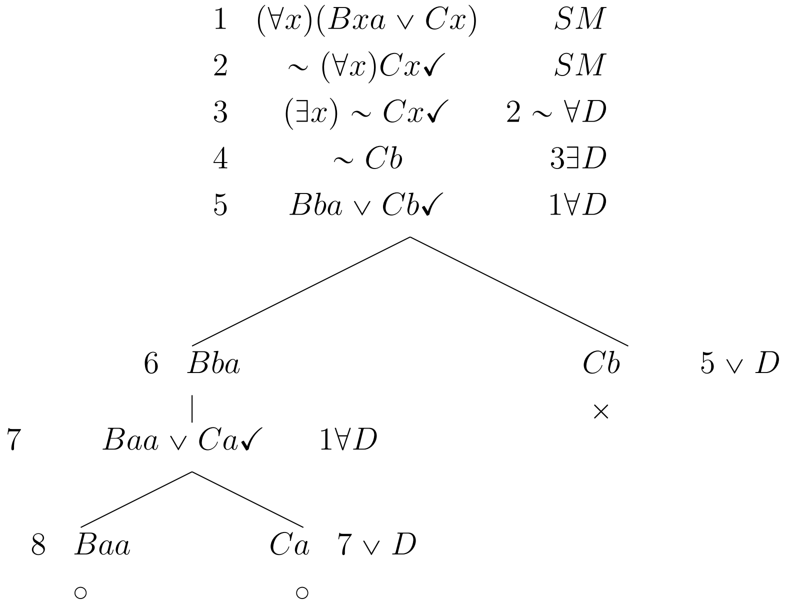
III. An **open branch** is a branch that is not closed.

IV. A **completed open branch** is an open branch such that each sentence on the branch is one of the following:

1. A literal
2. A decomposed (i.e. checked off) truth-functional or existentially quantified sentence.
3. A universally quantified sentence such that: (a) there is at least one substitution instance of the universally quantified sentence occurring on the branch and (b) there is a substitution instance of the universally quantified sentence on the branch corresponding to *each constant* that occurs on the branch.

We indicate a completed open branch by drawing a small circle underneath.

The left branch in the example from the previous section does not actually satisfy these criteria because we do not have a substitution instance of the universally quantified sentence for *each* constant occuring on the branch. However this is easily remedied in a few more steps.



V. A **closed tree** is a tree in which every branch is closed.

VI. An **open tree** is a tree that is not closed.

VII. A **completed tree** is a tree such that every branch is either closed or a completed open branch.

8.5 Strategies/Rules of thumb for working with truth trees

Here are some rules of thumb to use when working with truth trees:

1. Give priority to *non-branching* rules.
2. Give priority to sentences that will close one or more branches if decomposed.
3. Give priority to $\exists D$ over $\forall D$.
4. When using $\forall D$, use a constant that already occurs on the branch as your instantiating constant, unless no constants occur on the branch at all (in this case you may choose any constant).
5. Stop when you've answered the question that was asked of you.
6. If 1-5 are inapplicable, decompose more complex sentences first.

8.6 Determining Quantificational Properties

Recovering interpretations from completed open branches

Consider the following completed, open tree:

1	$\sim (\forall x)(\forall y)(Axy \supset Bxy)\checkmark$		SM
2	$(\exists x)(\exists y)(\sim Axy \vee \sim Bxy)\checkmark$		SM
3	$(\exists x) \sim (\forall y)(Axy \supset Bxy)\checkmark$		$1 \sim \forall D$
4	$\sim (\forall y)(Aay \supset Bay)\checkmark$		$3 \exists D$
5	$(\exists y)(\sim Aby \vee \sim Bby)\checkmark$		$2 \exists D$
6	$(\exists y) \sim (Aay \supset Bay)\checkmark$		$4 \sim \forall D$
7	$\sim (Aac \supset Bac)\checkmark$		$6 \exists D$
8	Aac		$7 \sim\supset D$
9	$\sim Bac$		$7 \sim\supset D$
10	$\sim Abd \vee \sim Bbd\checkmark$		$5 \exists D$
11	$\sim Abd$	$\sim Bbd$	$10 \vee D$
	○	○	

The tree has two completed open branches.

It can be proven (though we won't do it here) that it is *always* possible to construct an interpretation from a completed open branch that makes every sentence on the branch true.

In fact, it is easy to construct a distinct interpretation corresponding to each completed open branch of the tree.

(Note: a branch of a tree includes the "root" of the tree. So lines 1-10 belong to both the left and right branches in the example above).

Here is the procedure for constructing an interpretation from a branch:

1. Make a list containing each literal that occurs on that branch.
2. Create a UD that has exactly as many members as the number of constants occurring in this list.
3. Assign each member to a constant.
4. Interpret the predicates so that each literal turns out to be true.

The literals on the left branch are $Aac, \sim Bac, \sim Abd$.

Here is an interpretation that makes all of these true:

UD: {1, 2, 3, 4}
a: 1
b: 2
c: 3
d: 4
Axy: $x = y \div 3$
Bxy: $x = y$

This interpretation will make *every* sentence, not just the literals, true. Try it and see. Remember to keep in mind that your UD is *not* the set of positive integers. It is the set {1, 2, 3, 4}. It contains *only* those four numbers.

Let us consider the right branch. The literals on the right branch are $Aac, \sim Bac, \sim Bbd$.

Here is an interpretation that makes all of these true:[14]

---

[14] Note, in this case we could have used the same interpretation for both the left and right branch. But this won't always be possible in every case.

UD: {1, 2, 3, 4}
a: 1
b: 2
c: 3
d: 4
Axy: $x < y$
Bxy: $x > y$

Just like the left branch, this interpretation will make *every* sentence, not just the literals, true.

Relation between literals and set member sentences

The reason this procedure works is that the following relation holds between the literals on a completed open branch, and the set members (SMs) of the corresponding tree.

- If every literal on a completed open branch is true on an interpretation, then every set member is also true on that interpretation.

The opposite relation *does not* hold, however. If every set member is true on an interpretation, then it does not necessarily follow that every literal on a particular completed open branch is also true on that interpretation.

Consider the following tree:

$$\begin{array}{lcl} 1. & (\exists x)Px & SM \\ 2. & Pa & 1\exists D \\ & \circ & \end{array}$$

The following interpretation makes the existentially quantified sentence true and the literal false.

UD: {1, 2}
a: 2
Px: x is odd.

It is true that something in the UD is odd, since 1 is odd. However it is not true that 2 is odd.

Now of course it is the case that there is *some* interpretation on which $(\exists x)Px$ and Pa are both true (since if $(\exists x)Px$ is true then at least one thing is a P and all we have to do is make a represent one of those things). But in order to find such an interpretation, we must begin by interpreting the *literals* on the branch and not the set members—just as we have been doing.

8.6.1 Quantificational consistency and inconsistency

Recall that a set of sentences is q-consistent iff there is an interpretation on which every sentence in the set is true.

Now we just showed how, given a completed open branch, we can construct an interpretation that makes every sentence on the branch true.

But note that every branch of the tree will include the set members (SMs) of the tree. ($\{\sim (\forall x)(\forall y)(Axy \supset Bxy),\ (\exists x)(\exists y)(\sim Axy \vee \sim Bxy)\}$ in the example above). So these will be true as well on that interpretation.

So if a tree has a completed open branch, then we can say that the set consisting of the SMs is q-consistent, since we can construct an interpretation on which they are all true.

On the other hand, if every branch *closes*, then it is not possible to construct an interpretation on which every set member (SM) is true.

This brings us to the definitions of q-consistency and q-inconsistency for truth trees.

Quantificational Consistency A finite set Γ of sentences of PL is *quantificationally consistent* iff Γ does not have a closed truth tree.

Quantificational Inconsistency A finite set Γ of sentences of PL is *quantificationally inconsistent* iff Γ has a closed truth tree.

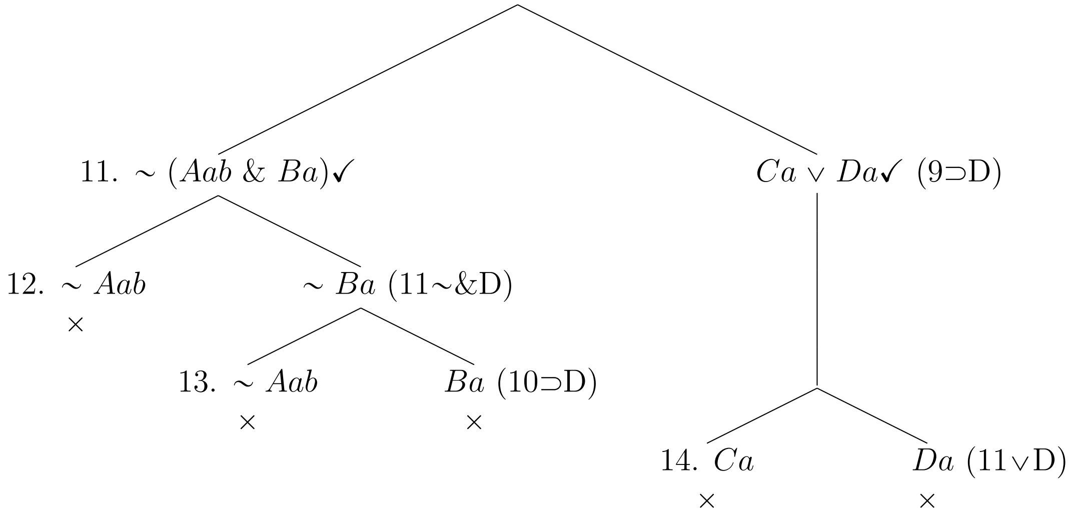
To determine whether a set of sentences, Γ, is consistent or inconsistent:

- Construct a tree in which each member of the set is one of the SMs (set members) of the tree.
- Does the tree close?
  - Yes: Γ is inconsistent.
  - No: Γ is consistent. Use one of the completed open branches of the tree to construct an interpretation on which every member of Γ is true.

Example

Determine whether the set $\Gamma = \{(\forall x)((Axb \ \& \ Bx) \supset (Cx \vee Dx)), (\exists x)(Axb \ \& \ ({\sim}\, Cx \ \& \ {\sim}\, Dx)),$ $(\forall x)(Axb \supset Bx)\}$ is q-consistent or q-inconsistent.

1. $(\forall x)((Axb \;\&\; Bx) \supset (Cx \vee Dx))$ (SM)
2. $(\exists x)(Axb \;\&\; ({\sim}\, Cx \;\&\; {\sim}\, Dx))$✓ (SM)
3. $(\forall x)(Axb \supset Bx)$ (SM)
4. $Aab \;\&\; ({\sim}\, Ca \;\&\; {\sim}\, Da)$✓ (2∃D)
5. Aab (4&D)
6. ${\sim}\, Ca \;\&\; {\sim}\, Da$✓ (4&D)
7. ${\sim}\, Ca$ (6&D)
8. ${\sim}\, Da$ (6&D)
9. $(Aab \;\&\; Ba) \supset (Ca \vee Da)$✓ (1∀D)
10. $Aab \supset Ba$✓ (3∀D)

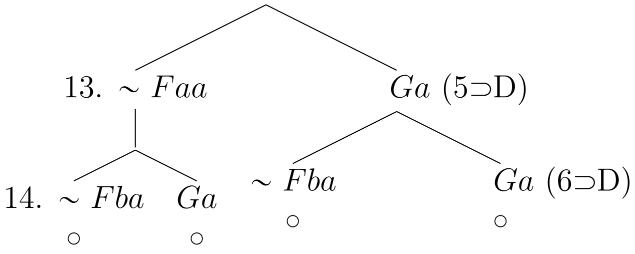


The tree is closed, so Γ is inconsistent.

Example

Determine whether the set $\Gamma = \{(\exists y)(\forall x)(Fxy \supset Gy), (\exists y)(\forall x)(Fxy \;\&\; {\sim}\, Gy)\}$ is q-consistent or q-inconsistent.

1. $(\exists y)(\forall x)(Fxy \supset Gy)\checkmark$ (SM)
2. $(\exists y)(\forall x)(Fxy \ \& \ \sim Gy)\checkmark$ (SM)
3. $(\forall x)(Fxa \supset Ga)$ (1∃D)
4. $(\forall x)(Fxb \ \& \ \sim Gb)$ (2∃D)
5. $Faa \supset Ga\checkmark$ (3∀D)
6. $Fba \supset Ga\checkmark$ (3∀D)
7. $Fab \ \& \ \sim Gb\checkmark$ (4∀D)
8. $Fbb \ \& \ \sim Gb\checkmark$ (4∀D)
9. Fab (7&D)
10. $\sim Gb$ (7&D)
11. Fbb (8&D)
12. $\sim Gb$ (8&D)



Let us recover an interpretation. We can choose any branch for this purpose. Let us choose the leftmost branch.

The literals on this branch are $Fab, \sim Gb, Fbb, \sim Faa, \sim Fba$.

We will use the following interpretation:

UD: {1, 2}
a: 1
b: 2
Fxy: x times y is greater than x

Gx: $x < 1$

8.6.2 Quantificational Falsity

To determine whether a sentence, **P**, is q-false:

- Construct a tree in which **P** is the only SM.
- Does it close?
  - Yes: then **P** is q-false.
  - No: then **P** is not q-false.

8.6.3 Quantificational Truth

To determine whether a sentence, **P**, is q-true:

- Construct a tree in which $\sim$ **P** is the only SM.
- Does it close?
  - Yes: then **P** is q-true (because $\sim$ **P** is q-false).
  - No: then **P** is not q-true.

8.6.4 Quantificational Indeterminacy

To determine whether a sentence, **P**, is q-indeterminate:

- Construct a tree in which **P** is the only SM.
- Does it close?
  - Yes: then **P** is q-false and hence *not* q-indeterminate.
  - No: then construct a tree in which $\sim$ **P** is the only SM.
  - Does it close?
    - Yes: then **P** is q-true, and hence *not* q-indeterminate.
    - No: then **P** is q-indeterminate.

8.6.5 Quantificational Equivalency

To determine whether the sentences **P** and **Q** are q-equivalent:

- Construct a tree in which $\sim (\mathbf{P} \equiv \mathbf{Q})$ is the only SM.
- Does it close?
  - Yes: then they are q-equivalent (since $\sim (\mathbf{P} \equiv \mathbf{Q})$ is q-false and hence $(\mathbf{P} \equiv \mathbf{Q})$ is q-true).
  - No: then they are not q-equivalent.

8.6.6 Quantificational Validity

To determine whether an argument,

1. Premise 1
2. Premise 2

∴ Conclusion

is q-valid, construct a tree in which all of the premises and the negation of the conclusion are the SMs of the tree:

1.	Premise 1	SM
2.	Premise 2	SM
3.	$\sim$ Conclusion	SM

- Does it close?
  - Yes: then the argument is q-valid.
  - No: then the argument is not q-valid.

8.6.7 Quantificational Entailment

To determine whether the set $\Gamma = \{\mathbf{S_1}, \mathbf{S_2}, \mathbf{S_3}, \mathbf{S_4}\} \models \mathbf{P}$:

Construct a tree in which all of the members of Γ and the negation of $\mathbf{P}$ are the SMs of the tree:

1. $\mathbf{S_1}$ SM
2. $\mathbf{S_2}$ SM
3. $\mathbf{S_3}$ SM
4. $\mathbf{S_4}$ SM
5. $\sim \mathbf{P}$ SM

- Does it close?
  - Yes: then $\Gamma \models \mathbf{P}$
  - No: then $\Gamma \not\models \mathbf{P}$

Example (note: from now on I will be writing the decomposition rules separately below the tree. This is due to my having inappropriate software for constructing the trees. When you construct trees, however, you should indicate the rules to the side of the tree as in the textbook).

Determine whether $\{(\forall x)(\forall y)((Axy \ \& \ Bya) \supset Cxay), (\exists x)(\forall y)Axy, (\forall x)(\forall y)(Axy \equiv Bya)\}$ $\models (\exists x)(\exists y)Cxay$

1. $(\forall x)(\forall y)((Axy \ \& \ Bya) \supset Cxay)$

2. $(\exists x)(\forall y)Axy$✓

3. $(\forall x)(\forall y)(Axy \equiv Bya)$

4. $\sim (\exists x)(\exists y)Cxay$✓

5. $(\forall y)Aby$

6. $(\forall x) \sim (\exists y)Cxay$

7. $(\forall y)((Aby \ \& \ Bya) \supset Cbay)$

8. $(Abb \ \& \ Bba) \supset Cbab$✓

9. $(\forall y)(Aby \equiv Bya)$

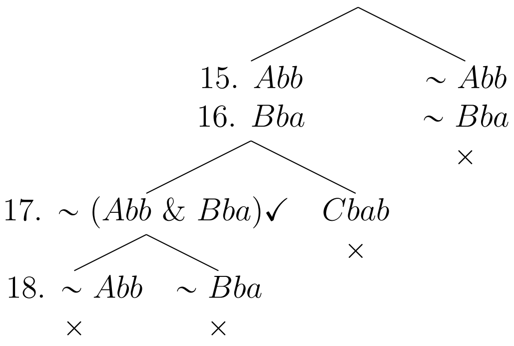
10. $Abb \equiv Bba$✓

11. Abb

12. $\sim (\exists y)Cbay$✓

13. $(\forall y) \sim Cbay$

14. $\sim Cbab$



Justification:

1. SM
2. SM
3. SM
4. SM
5. 2∃D
6. 4∼ ∃D
7. 1∀D
8. 7∀D
9. 3∀D
10. 9∀D
11. 5∀D
12. 6∀D
13. 12∼ ∃D
14. 13∀D
15. 10≡D
16. 10≡D
17. 8⊃D
18. 17&D

8.7 A new rule for existential decomposition (∃D2)

Consider the set with the single member, $\{(\forall x)(\exists z)Fyz\}$. One finite model (i.e. interpretation with a finite UD) for this set is the following:[15]

UD: {1,2,3}
Fxy: $\{\langle 1,1\rangle, \langle 2,1\rangle, \langle 3,1\rangle\}$

As we have already seen, one way to help us construct a finite model is to use the truth tree method: we construct a truth tree from all of the (in this case only one) sentences in the set. We then use one of the completed open branches to construct our finite model.

So let's try this for the set $\{(\forall x)(\exists z)Fyz\}$:

1. $(\forall y)(\exists z)Fyz$ (SM)
2. $(\exists z)Faz$✓ (1∀D)
3. Fab (2∃D)
4. $(\exists z)Fbz$✓ (1∀D)
5. Fbc (4∃D)
6. $(\exists z)Fcz$✓ (1∀D)
7. Fcd (6∃D)

.
.
.

---

[15] If the notation here seems strange to you, recall the definition of a predicate in terms of its extension from section 6.1.2 of these notes.

This branch will never complete. Remember the criterion for a completed open branch: each sentence must be one of:

1. A literal
2. A decomposed (i.e. checked off) truth-functional or existentially quantified sentence.
3. A universally quantified sentence such that: (a) there is at least one substitution instance of the universally quantified sentence occurring on the branch and (b) there is a substitution instance of the universally quantified sentence on the branch corresponding to *each constant* that occurs on the branch.

Every time we decompose the universally quantified sentence at line 1, it results in an existentially quantified sentence. When we decompose the existentially quantified sentence, we must use a constant foreign to the branch as the instantiating constant. But by adding a new constant to the branch, we make it so that criterion 3 is not satisfied. So we must decompose line 1 again, which results in yet another existentially quantified sentence, and then in yet another new constant in turn. And so on. It is easy to see that this process will continue forever.

We call this branch a **non-terminating** branch. Although it is open, it is not, and never will be, a **completed** open branch. But we cannot use the tree to find a finite model for this set unless we have a completed open branch.

Obviously there is a finite model for this set, for we constructed such a finite model at the beginning of this section. But there does not seem to be a way, using the rules we have defined so far, to discover such a finite model using a tree, even though we know that one exists.

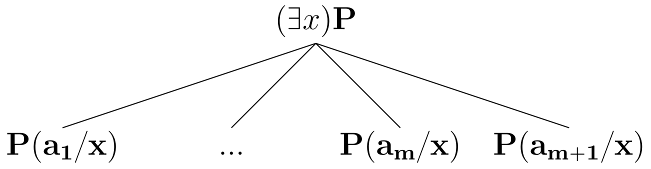
One way to remedy this is to think about existential decomposition in a different way.

When presenting the constraint of ∃D that the instantiating constant be foreign to the branch, we justified this by pointing out that it is illegitimate to presume that the member of the UD that satisfies the formula **P** in a sentence such as $(\exists x)\mathbf{P}$ just so happens to be one for which you already have some knowledge (this knowledge is represented by the statements on the branch so far).

But another way to think about things that still captures this intuition is the following. If we know that *something* satisfies the formula **P**, then that something could be a thing that we already know something about, or it could be some other thing. In this way we do not presume *that it is* something that we already know something about (so there is no violation of the intuition behind ∃D that we mentioned above). But at the same time we do allow that *it could be.*

So let us create a new rule. We will be extra-imaginative and call it ∃D2. To represent the fact that the member of the UD that satisfies **P** could be a thing that we already know something about, or that it could be some other thing, we will make the rule a **branching** rule:

∃D2:



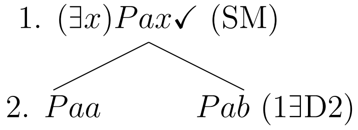
The rule will work as follows. When decomposing an existentially quantified sentence on a particular branch:

- If that branch currently has 0 constants on it, then ∃D2 works the same as ∃D.
- If that branch currently has 1 constant on it, then we create substitution instances for the existentially quantified sentence on 2 new branches. The first branch will use, for its instantiating constant, the constant we already have on the old branch. The second will contain a new constant.
- If that branch currently has 2 constants on it, then we create substitution instances for the existentially quantified sentence on 3 new branches. The first branch will use, for its instantiating constant, the first constant from the old branch. The second branch will use, for its instantiating constant, the second constant from the old branch. The third branch will use a new constant.
- Similarly 3 constants will yield 4 branches, 4 constants will yield 5 branches, and so on. In general for a branch with m constants present, you must create $m + 1$ new branches.

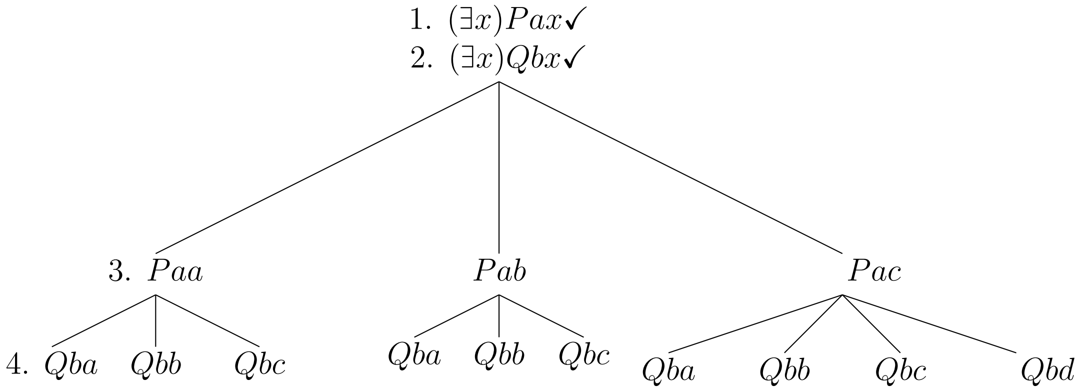
Example 1

1. $(\exists x)Px$✓ (SM)
2. Pa (1∃D2)

Example 2



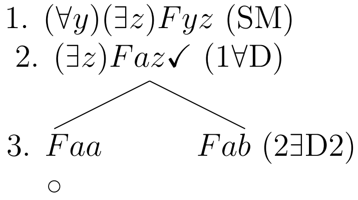
Example 3



Justification for example 3:

1. SM
2. SM
3. 1∃D2
4. 2∃D2

Let us now consider again the example that we started this section with and see if we can get a completed open branch using our new rule:



Yes we can. And we can construct the following interpretation based on it:

UD: {1}
a: 1

Fxy: x = y, or perhaps: Fxy: $\{\langle 1, 1\rangle\}$

8.8 Systematic Trees

8.8.1 The non-systematic tree method

The tree method we have been using so far is called the **non-systematic method**.

When constructing non-systematic trees, you may use either ∃D, ∃D2, or (if you insist on confusing yourself) both.

The advantage to using ∃D is that the resulting tree will generally be shorter.

On the other hand, a disadvantage is that if a finite model exists for a set of sentences, then, as we have seen, it will not always be possible to find it using ∃D. To guarantee that it is possible to find a finte model for a set of sentences if one exists, we must use ∃D2.

However, just because it is possible to find a finite model does not mean we will actually manage to find it. The strategies or "rules of thumb" that we mentioned earlier are useful guides for constructing non-systematic trees, but they do not constitute a sure-fire method for finding a completed open branch.

The case is similar for sets of sentences that are inconsistent. Using the non-systematic method and our "rules of thumb" will not guarantee that the tree will actually close.

Even if they were a sure-fire method, since they are only "rules of thumb", there is nothing stopping us from not following them. It is a rule of thumb, for instance, that we should decompose a universally quantified sentence with an instantiating constant that already occurs on the branch, but there is nothing stopping us from using any old constant. For instance, here is an extremely silly—but perfectly legitimate—way of decomposing the tree made up from $\{(\forall x)Fx, (\forall x) \sim Fx\}$:

1. $(\forall x)Fx$ (SM)
2. $(\forall x) \sim Fx$ (SM)
3. Fa (1∀D)
4. Fb (1∀D)
5. Fc (1∀D)

.
.
.

Obviously it is *possible* to close this tree in just a couple of steps. But it is equally obvious that if we continue in this very silly way we will never close the tree. This is a silly example. But there are less silly examples that illustrate the same point.

To guarantee that **we will** get a completed open branch on the tree for a set of sentences with a finite model, or that a tree **will** close for a set of inconsistent sentences, we must use **the systematic method.**

8.8.2 The Systematic Method

This is how we use the systematic method:

- **Always use ∃D2**, never ∃D.
- **Exit conditions**
  - The tree closes
  - A branch of the tree becomes a completed open branch.
- **Stage 1:** Decompose *all* truth-functional compounds and *all* existentially quantified sentences (including new ones that result from the process of decomposition).
- **Stage 2:** For each universally quantified sentence, decompose it on *every* branch that it is applicable to, for *every* constant on that branch. If there are no constants on a branch to which it is applicable, then decompose it once.
- **Go back to stage 1 and repeat** until you get one of the exit conditions.

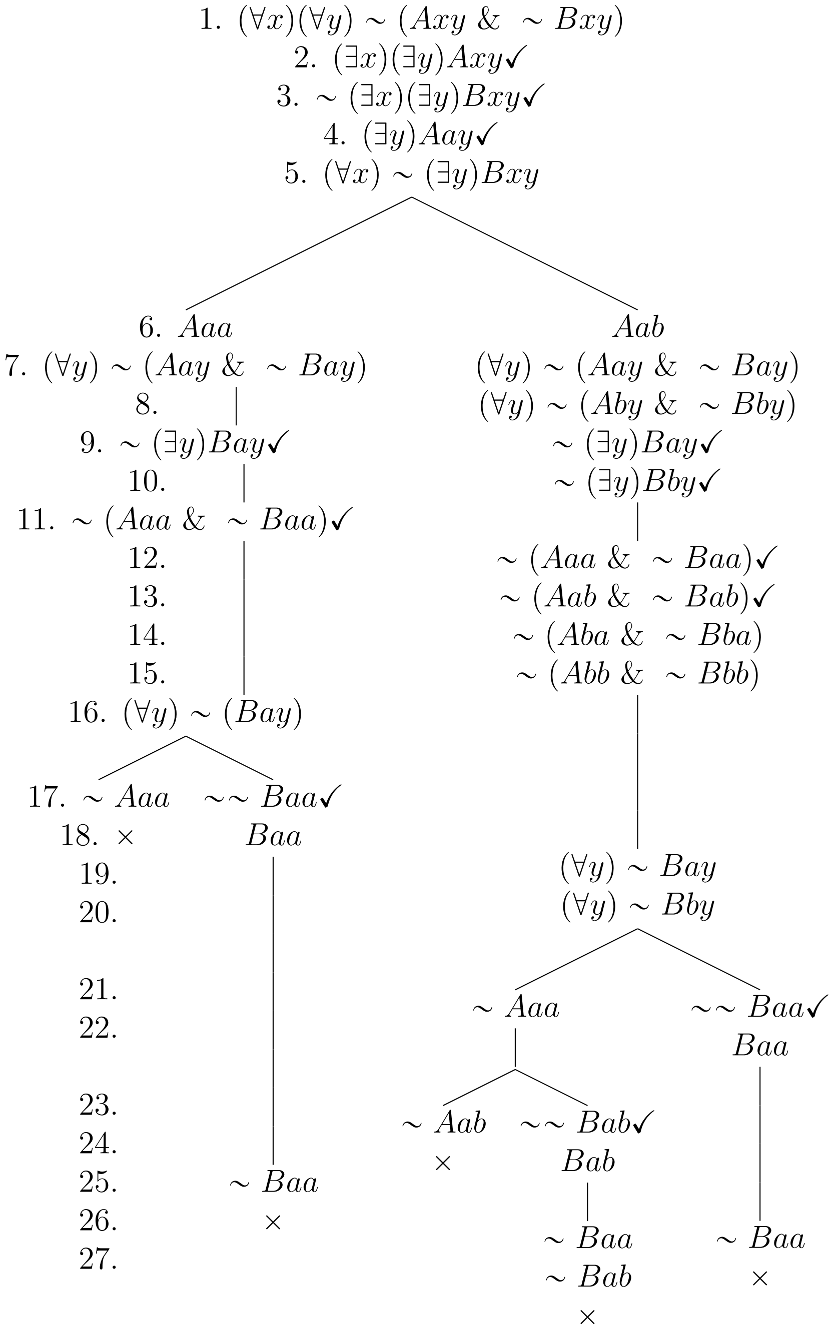
Example

Here is an example of a systematic tree designed to show that the argument:

1. $(\forall x)(\forall y) \sim (Axy \;\&\; \sim Bxy)$
2. $(\exists x)(\exists y)Axy$

$\therefore\ (\exists x)(\exists y)Bxy$

is q-valid. Note how we go from stage 1 to stage 2 then back to stage 1 and then back to stage 2. Remember you must *complete* stage 1 in its entirety before moving on to stage 2, and vice versa.


1. $(\forall x)(\forall y) \sim (Axy \;\&\; \sim Bxy)$
2. $(\exists x)(\exists y)Axy$✓
3. $\sim (\exists x)(\exists y)Bxy$✓
4. $(\exists y)Aay$✓
5. $(\forall x) \sim (\exists y)Bxy$
6. Aaa
7. $(\forall y) \sim (Aay \;\&\; \sim Bay)$
8.
9. $\sim (\exists y)Bay$✓
10.
11. $\sim (Aaa \;\&\; \sim Baa)$✓
12.
13.
14.
15.
16. $(\forall y) \sim (Bay)$
17. $\sim Aaa$
$\sim\sim Baa$✓
18. ×
Baa
19.
20.
21.
22.
23.
24.
25.
$\sim Baa$
26.
×
27.
Aab
$(\forall y) \sim (Aay \;\&\; \sim Bay)$
$(\forall y) \sim (Aby \;\&\; \sim Bby)$
$\sim (\exists y)Bay$✓
$\sim (\exists y)Bby$✓
$\sim (Aaa \;\&\; \sim Baa)$✓
$\sim (Aab \;\&\; \sim Bab)$✓
$\sim (Aba \;\&\; \sim Bba)$
$\sim (Abb \;\&\; \sim Bbb)$
$(\forall y) \sim Bay$
$(\forall y) \sim Bby$
$\sim Aaa$
$\sim\sim Baa$✓
Baa
$\sim Aab$
$\sim\sim Bab$✓
×
Bab
$\sim Baa$
$\sim Bab$
×
$\sim Baa$
×

Justification

1. SM
2. SM
3. SM
4. 2∃D2
5. 3∼ ∃D
6. 4∃D2
7. 1∀D
8. 1∀D
9. 5∀D
10. 5∀D
11. 7∀D
12. 7∀D
13. 7∀D
14. 8∀D
15. 8∀D
16. 9∼ ∃D
17. 11∼&D
18. 17∼∼D
19. 9∼ ∃D

20. 10$\sim\exists$D

21. 12$\sim$&D

22. 21$\sim\sim$D

23. 13$\sim$&D

24. 23$\sim\sim$D

25. 16$\forall$D

26. 19$\forall$D

27. 19$\forall$D

The systematic method will *always* give you a completed open branch when the set of sentences has a finite model, and it will *always* give you a closed tree when the set of sentences is inconsistent.

However the systematic method *will not* result in a completed open branch for sets that have *only* infinite models, i.e. for sets that can only be interpreted using an infinite UD (an example of an infinite UD is the set of positive integers, {1, 2, 3, 4, ...}). In cases like these, the tree will *never* complete, i.e., it will go on forever.

For a set of sentences that *only* has infinite models, one cannot use a tree to construct an interpretation. One can only construct one directly. For instance, the set of sentences $\{(\forall x)(\exists y)Fxy, \sim(\exists x)Fxx, (\forall x)(\forall y)(\forall z)[(Fxy \;\&\; Fyz) \supset Fxz]\}$ has only infinite models. Here is one such model:

UD: positive integers
Fxy: $x < y$

In general you will not know beforehand whether or not a set of sentences has only infinite models. So if you are constructing your tree, how will you know when to give up?

In general you will not know. However if you have gone through each stage in the systematic method a few times and are still no closer to completing the tree, you should consider the possibility that the set of sentences has only infinite models and consider abandoning the tree.

8.8.3 Summary

A **model** for a set of sentences $\Gamma = \{\mathbf{P}, \mathbf{Q}, \mathbf{R}, ...\}$ is an interpretation on which each of **P, Q, R, ...** is true.

A **finite** model is a model with a finite UD; e.g, {1, 2, 3}.

An **infinite model** is a model with an infinite UD; e.g., positive integers.

Now for any set of sentences $\Gamma = \{\mathbf{P}, \mathbf{Q}, \mathbf{R}, ...\}$, there are 3 possibilities.

I. Γ has only infinite models. In this case,

- You will not be able to construct any of these models using a truth-tree. (You must construct a model without a tree.)

II. Γ has some finite models. In this case,

- If you use the **non-systematic method and ∃D**, there is no guarantee that it will be possible to construct a finite model based on such a tree.
- **If you use the non-systematic method and ∃D2**, then it is guaranteed that it will be *possible* to construct a finite model for the set based on such a tree, but there is no guarantee that you will succeed.
- **If you use the systematic method**, it is guaranteed that you will succeed in constructing a finite model for the set based on such a tree.

III. Γ is inconsistent. In this case,

- **The non-systematic method** will not guarantee that the tree will close.
- **If you use the systematic method**, the tree is guaranteed to close.